Linear Algebra

Book 3

Eg. $A = \begin{bmatrix} 1 & 3 & 0 & 41 \\ 2 & 4 & 11 & 7 \\ 0 & 3 & 0 & 4 \\ 1 & 6 & 3 & 5 \end{bmatrix}$

Expanding along the third row, $\det A = \cancel{0 \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 3 \begin{vmatrix} 1 & 0 & 41 \\ 2 & 11 & 7 \\ 1 & 3 & 5 \end{vmatrix} + \cancel{0 \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 4 \begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 11 \\ 1 & 6 & 3 \end{vmatrix}$

$$= -3\left(\begin{vmatrix} 11 & 7 \\ 3 & 5 \end{vmatrix} + 41 \begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix} \right) - 4 \left(\begin{vmatrix} 4 & 11 \\ 6 & 3 \end{vmatrix} - 3 \begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix} \right)$$

$$= -3\left(55 - 21 + 41(6-11)\right) - 4\left(12 - 66 - 3(6-11)\right)$$

$$= 669.$$

(I checked this by computer.)

Wed. Nov. 8 Test. Come a few minutes early if you can.
No Quiz Fri. Nov 10, 17.
I am away Fri. Nov. 17, Mon Nov 20. Lectures for those two days will be prerecorded - check the websites.

Recall: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = ad - bc$. A is invertible iff $\det A \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This formula has a generalization for $n \times n$ matrices (Cramer's Rule). This is useful although not the most computationally efficient way to compute A^{-1} if n is large.

On HW 2 you had to find A^{-1} where A is 4×4. The entries of A^{-1} have a common denominator $6 = \det A$.

Eg. $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix}$, $\det A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -3 & -7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -8 & -31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & 8 & 31 \end{vmatrix} = \left|\begin{array}{c|cc} 1 & 2 & 5 \\ \hline 0 & 3 & 7 \\ 0 & -1 & 10 \end{array}\right|$

$$= |1| \begin{vmatrix} 3 & 7 \\ -1 & 10 \end{vmatrix} = 1 \cdot 37.$$

A^{-1} has fractional entries with common denominator 37.

Matrix of minors : $M = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -14 & -13 & 5 \\ -22 & -31 & -8 \\ 1 & -7 & -3 \end{bmatrix}$

$A^{-1} = \frac{1}{37} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix}$

transpose ;
apply checkerboard ;
divide by det A

$$= \begin{bmatrix} -\frac{14}{37} & \frac{22}{37} & \frac{1}{37} \\ \frac{13}{37} & -\frac{31}{37} & \frac{7}{37} \\ \frac{5}{37} & \frac{8}{37} & -\frac{3}{37} \end{bmatrix}$$

Check: $A A^{-1} = \frac{1}{37} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 37 & 0 & 0 \\ 0 & 37 & 0 \\ 0 & 0 & 37 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If A is a square matrix with integer entries and $\det A = \pm 1$, then A^{-1} also has integer entries.

Find a constant c such that the following matrix has determinant zero:

$$A = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 2 & 4 \\ 7 & 7 & c \end{bmatrix} \begin{matrix} \leftarrow u = (5\ 3\ 6) \\ \leftarrow v = (1\ 2\ 4) \\ \leftarrow w \quad u + 2v = (7\ 7\ 14) \end{matrix}$$

If $c = 14$ then A has linearly dependent rows so $\det A = 0$ in this case (A is not invertible).

If $c \neq 14$ then A has linearly independent rows then $w \neq (7\ 7\ 14)$ and $(0\ 0\ 1)$ is a linear combination of u, v, w i.e. Row A contains $u, v, (0\ 0\ 1)$.

$$\det \left[\begin{array}{cc|c} 5 & 3 & 6 \\ 1 & 2 & 4 \\ \hline 0 & 0 & 1 \end{array}\right] = 7 \times 1 = 7 \neq 0 .$$