Linear Algebra

Book 2

$f: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$f(x,y) = (3x+2y\ ,\ x-5y)$ can be represented as a matrix transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{bmatrix} 3 & 2 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+2y \\ x-5y \end{bmatrix}$$

Every linear operator can be expressed as matrix multiplication

eg. consider solutions of $y''+y=0$ i.e. $f(x) = \underbrace{a \sin x + b \cos x}_{\begin{bmatrix} a \\ b \end{bmatrix}}$

$$Df(x) = \underbrace{a\cos x - b \sin x}_{\begin{bmatrix} -b \\ a \end{bmatrix}}$$

$$D(rf+sg) = rDf + sDg$$

$$(rf+sg)' = rf' + sg'$$

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{M}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$$

$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$M^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$M^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$M^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Every 2×2 real matrix A represents a linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ which is the matrix transformation $T_A\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}x\\y\end{bmatrix}$.

eg. $\underbrace{\begin{bmatrix}0&-1\\1&0\end{bmatrix}}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}-y\\x\end{bmatrix}$ T_A is a counter-clockwise 90° rotation about the origin in $\mathbb{R}^2$:

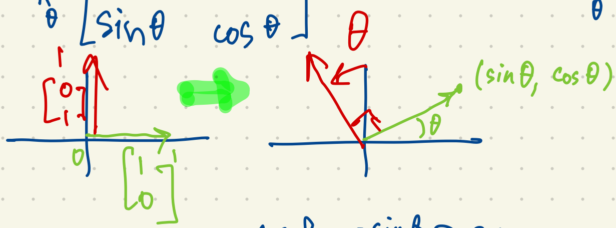
$$T_A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T_A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0&-1\\1&0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$T_A^4 = I \qquad I\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

A counterclockwise rotation by angle θ about the origin in $\mathbb{R}^2$ represented by the matrix $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$

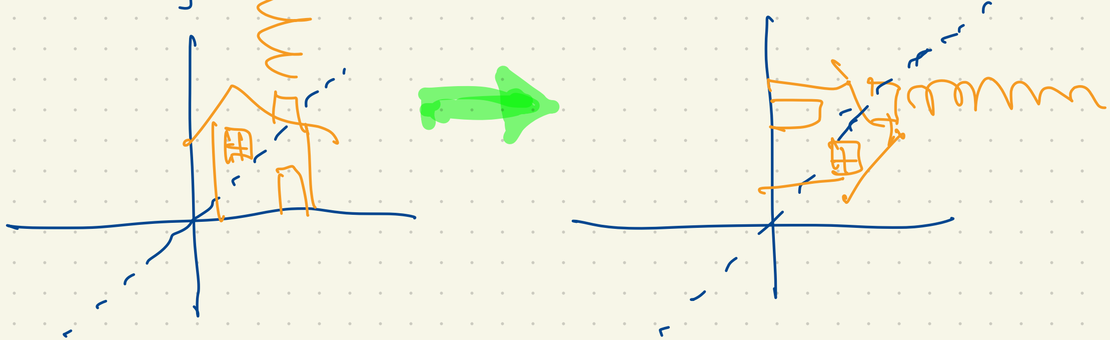
$$R_\theta\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} \qquad R_\theta\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$



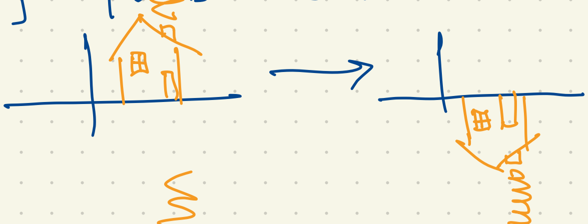
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$R_\beta R_\alpha = R_{\alpha+\beta} \qquad \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}$$

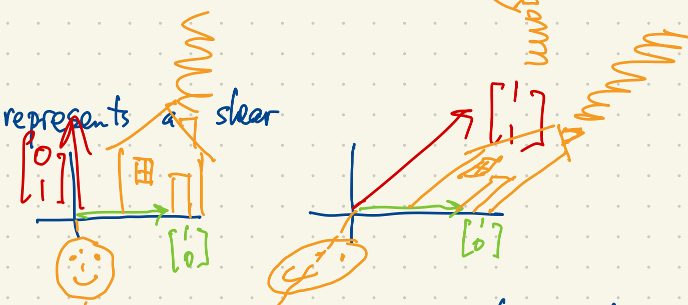
Eg. $\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}y\\x\end{bmatrix}$ is a reflection about the line $y=x$



$\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ represents a reflection in the x-axis

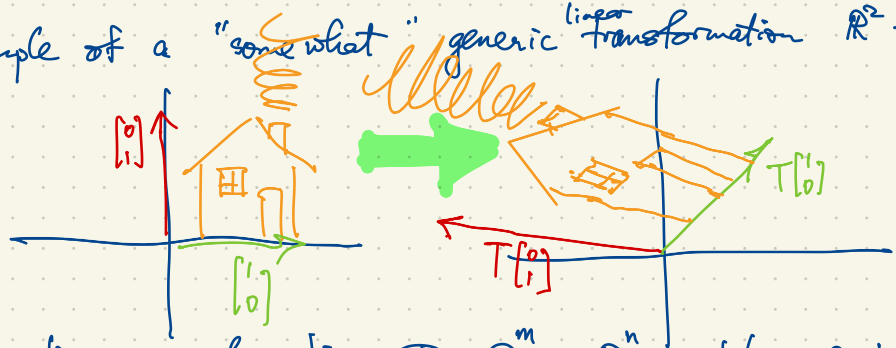


$\begin{bmatrix}1&1\\0&1\end{bmatrix}$ represents a shear



Every matrix transformation is a linear transformation: it takes 0 to 0 and it takes lines to lines or points. It may distort distances and angles.

Example of a "somewhat" generic linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$:



Every linear transformation $T: \mathbb{R}^m \to \mathbb{R}^n$ takes 0 to 0,

takes lines to lines or points