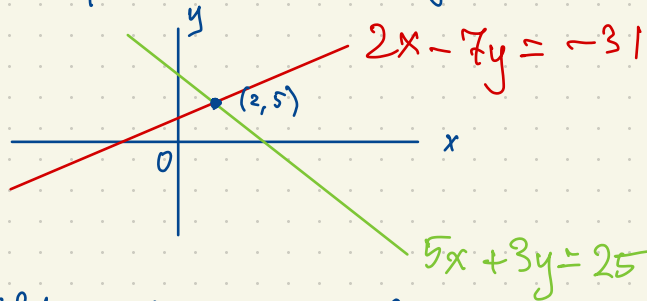


Linear Algebra

Book 1

Example: Find all (x, y) such that $\underline{5x+3y=25}$ and $\underline{2x-7y=-31}$.



We are asking for the simultaneous solution of a system of two equations in two unknowns x and y .

$$\begin{cases} 5x + 3y = 25 & (1) \\ 2x - 7y = -31 & (2) \end{cases}$$

$$\begin{aligned} 41y &= 205 \\ y &= 5 \end{aligned}$$

$$\begin{aligned} 5x + 15 &= 25 \\ 5x &= 10 \\ x &= 2 \end{aligned}$$

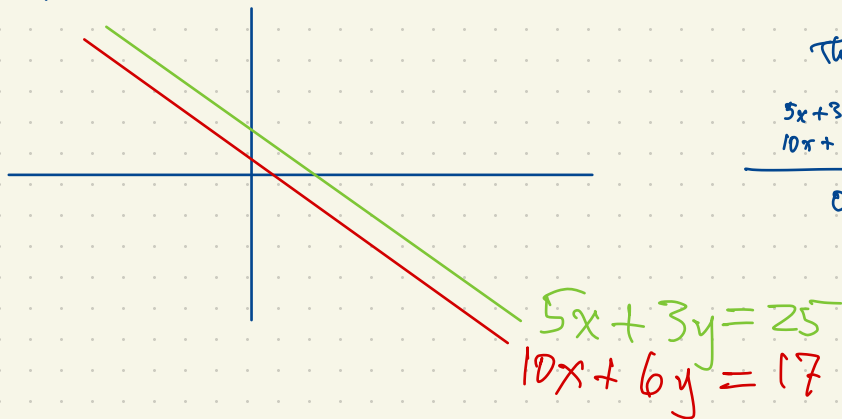
$$\begin{aligned} 2 \times (1) - 5 \times (2) &= (3) \\ (1) &= (3) \div 41 \end{aligned}$$

$$2 \times 3 - 5(-7) = 6 + 35 = 41$$

$$2 \times 25 - 5 \times (-31) = 50 + 155 = 205$$

Solution: $(x, y) = (2, 5)$ is the unique solution.

Example: Find all (x, y) such that $\underline{5x+3y=25}$ and $\underline{10x+6y=17}$.



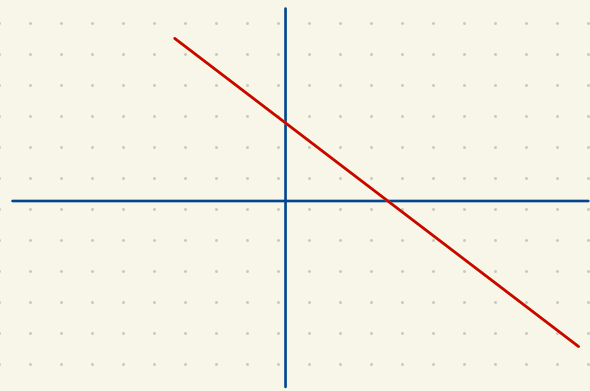
This system is inconsistent: it has no solution.

$$\begin{cases} 5x + 3y = 25 & (1) \\ 10x + 6y = 17 & (2) \end{cases}$$

$$0 = 33 \quad 2 \times (1) - (2)$$

This is inconsistent.

Example: Find all (x, y) such that $5x + 3y = 25$ and $15x + 9y = 75$.



This system is consistent but the solution is not unique: there are infinitely many solutions.

$$\begin{array}{rcl} 5x + 3y = 25 & (1) \\ 15x + 9y = 75 & (2) \\ \hline 0 = 0 & (3) = 3 \times (1) - (2) \end{array}$$

$$5x + 3y = 25$$
$$15x + 9y = 75$$

A system of m linear equations in n unknowns has the form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

$(a_{ij}, b_i$ constants for $i \in \{1, \dots, m\}$, $j \in \{1, 2, \dots, n\}$; x_1, \dots, x_n variables representing unknowns).

Typically, when $m = n$ we can expect a unique solution;
 $m > n$: - - - no solution (inconsistent system);
 $m < n$: - - - more than one solution.

Example with $m=n=3$: a system of 3 linear equations in 3 unknowns.
 Kim buys a bag of 26 items weighing 226 oz. costing \$34. The items included

cans of tuna (\$1 each, 5oz each)

apples (\$1 each, 8oz each)

loaves of bread (\$3 each, 20oz each)

How many of each item did Kim buy? (say x cans of tuna, y apples, z loaves of bread)

$$x + y + z = 26 \quad (1)$$

$$5x + 8y + 20z = 226 \quad (2)$$

$$x + y + 3z = 34 \quad (3)$$

$$2z = 8 \quad (3) - (1) = (4)$$

$$z = 4 \quad (5)$$

$$x + y = 22 \quad (6) = (8) - (5)$$

$$5x + 8y = 146 \quad (7)$$

$$3y = 36 \quad (7) - 5 \times (6) = (8)$$

$$y = 12 \quad (9) = (8) \div 3$$

$$x = 10 \quad (10) = (6) - (9)$$

$$146 - 5 \times 22 = 146 - 110 = 36$$

The unique solution of this system is $(x, y, z) = (10, 12, 4)$.

(Kim bought 10 cans of tuna, 12 apples, and 4 loaves of bread.)

Check! that all three equations are satisfied.

Matrix formulation of linear systems

$$\begin{aligned} x + y + z &= 26 \\ 5x + 8y + 20z &= 226 \\ x + y + 3z &= 34 \end{aligned} \quad \rightarrow \quad \begin{array}{ccc|c} x & y & z & \text{total} \\ \hline 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{array}$$

$$\begin{bmatrix} 1 & 1 & 1 & 26 \\ 5 & 8 & 20 & 226 \\ 1 & 1 & 3 & 34 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 0 & 2 & 8 \\ 0 & 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 2 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 3 & 15 & 96 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 1 & 5 & 32 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

subtract row 1 from row 3
divide row 3 by 2
subtract 5 times row 1 from row 2
divide row 2 by 3

$$226 - 5 \times 26 = 226 - 130 = 96$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 26 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 14 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad \text{i.e.} \quad \begin{aligned} x &= 10 \\ y &= 12 \\ z &= 4 \end{aligned}$$

subtract 5 times row 3 from row 2
subtract row 2 from row 1
subtract row 3 from row 1

Example: Find all (x, y) such that $5x + 3y = 25$ and $2x - 7y = -31$.

$$\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & -\frac{41}{5} & -41 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3}{5} & 5 \\ 0 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

divide row 1 by 5
subtract 2 times row 1 from row 2
multiply row 2 by $-\frac{5}{41}$
subtract $\frac{3}{5}$ times row 2 from row 1

$$-7 - \frac{6}{5} = -\frac{35}{5} - \frac{6}{5} = -\frac{41}{5}$$

$$-31 - 10 = -41$$

Solution: $(x, y) = (2, 5)$.

Alternatively:

$$\begin{bmatrix} 5 & 3 & 25 \\ 2 & -7 & -31 \end{bmatrix} \sim \begin{bmatrix} 2 & -7 & -31 \\ 5 & 3 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{7}{2} & -\frac{31}{2} \\ 5 & 3 & 25 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

interchange rows 1 and 2

Even better: $\begin{bmatrix} 5 & 3 & | & 25 \\ 2 & -7 & | & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & | & 87 \\ 2 & -7 & | & -31 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & | & 87 \\ 0 & -41 & | & -205 \end{bmatrix} \sim \begin{bmatrix} 1 & 17 & | & 87 \\ 0 & 1 & | & 5 \end{bmatrix}$

subtract 2 times
row 2
from row 1

subtract 2 times
row 1 from row 2

divide row 2
by -41

$$\begin{aligned}
 & -31 - 2 \times 87 \\
 & = -31 - 174 \\
 & = -205
 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 5 \end{bmatrix}$$

subtract 17 times row 2
from row 1

Solution: $(x, y) = (2, 5)$.

Check!

$$\begin{aligned}
 5 \times 2 + 3 \times 5 & = 25 \\
 2 \times 2 - 7 \times 5 & = -31
 \end{aligned}$$