Linear Algebra

Book 3

Eg. $A = \begin{bmatrix} 1 & 3 & 0 & 41 \\ 2 & 4 & 11 & 7 \\ 0 & 3 & 0 & 4 \\ 1 & 6 & 3 & 5 \end{bmatrix}$

Expanding along the third row, $\det A = \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 3\begin{vmatrix} 1 & 0 & 41 \\ 2 & 11 & 7 \\ 1 & 3 & 5 \end{vmatrix} + \cancel{0\begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}} - 4\begin{vmatrix} 1 & 3 & 0 \\ 2 & 4 & 11 \\ 1 & 6 & 3 \end{vmatrix}$

$$= -3\left(\begin{vmatrix} 11 & 7 \\ 3 & 5 \end{vmatrix} + 41\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right) - 4\left(\begin{vmatrix} 4 & 11 \\ 6 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 11 \\ 1 & 3 \end{vmatrix}\right)$$

$$= -3\left(55 - 21 + 41(6-11)\right) - 4\left(12 - 66 - 3(6-11)\right)$$

$$= 669.$$

(I checked this by computer.)

Wed. Nov. 8 Test. Come a few minutes early if you can.

No Quiz Fri. Nov 10, 17.

I am away Fri. Nov. 17, Mon Nov 20. Lectures for those two days will be prerecorded - check the websites.

Recall: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\det A = ad - bc$. A is invertible iff $\det A \neq 0$, in which case $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

This formula has a generalization for $n \times n$ matrices (Cramer's Rule). This is useful although not the most computationally efficient way to compute A^{-1} if n is large.

On HW 2 you had to find A^{-1} where A is 4×4. The entries of A^{-1} have a common denominator $6 = \det A$.

Eg. $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix}$, $\det A = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & -3 & -7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 7 & 6 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & -8 & -31 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & 7 \\ 0 & 8 & 31 \end{vmatrix} = \left|\begin{array}{c|cc} 1 & 2 & 5 \\ \hline 0 & 3 & 7 \\ 0 & -1 & 10 \end{array}\right|$

$$= |1| \begin{vmatrix} 3 & 7 \\ -1 & 10 \end{vmatrix} = 1 \cdot 37.$$

A^{-1} has fractional entries with common denominator 37.

Matrix of minors : $M = \begin{bmatrix} \begin{vmatrix} 1 & 3 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 7 & 6 \end{vmatrix} \\ \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 5 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -14 & -13 & 5 \\ -22 & -31 & -8 \\ 1 & -7 & -3 \end{bmatrix}$

$$A^{-1} = \frac{1}{37} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix}$$

transpose ;
apply checkerboard ;
divide by det A

$$= \begin{bmatrix} -\frac{14}{37} & \frac{22}{37} & \frac{1}{37} \\ \frac{13}{37} & -\frac{31}{37} & \frac{7}{37} \\ \frac{5}{37} & \frac{8}{37} & -\frac{3}{37} \end{bmatrix}$$

Check: $A A^{-1} = \frac{1}{37} \begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 3 \\ 7 & 6 & 4 \end{bmatrix} \begin{bmatrix} -14 & 22 & 1 \\ 13 & -31 & 7 \\ 5 & 8 & -3 \end{bmatrix} = \frac{1}{37} \begin{bmatrix} 37 & 0 & 0 \\ 0 & 37 & 0 \\ 0 & 0 & 37 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✓

If A is a square matrix with integer entries and $\det A = \pm 1$, then A^{-1} also has integer entries.

Find a constant c such that the following matrix has determinant zero:

$$A = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 2 & 4 \\ 7 & 7 & c \end{bmatrix} \begin{array}{l} \leftarrow u = (5\ 3\ 6) \\ \leftarrow v = (1\ 2\ 4) \\ \leftarrow w \quad u + 2v = (7\ 7\ 14) \end{array}$$

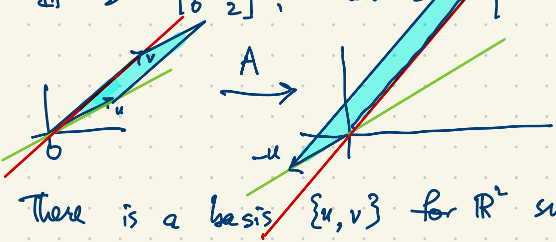
If $c = 14$ then A has linearly dependent rows so $\det A = 0$ in this case (A is not invertible).

If $c \neq 14$ then A has linearly independent rows then $w \neq (7\ 7\ 14)$
and $(0\ 0\ 1)$ is a linear combination of u, v, w i.e. Row A contains $u, v, (0\ 0\ 1)$.

$$\det \left[\begin{array}{cc|c} 5 & 3 & 6 \\ 1 & 2 & 4 \\ \hline 0 & 0 & 1 \end{array}\right] = 7 \times 1 = 7 \neq 0 .$$

If $A = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix}$, then $A^{10} =$

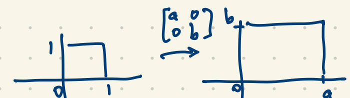
If $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, then $D^{10} = \begin{bmatrix} & \\ & \end{bmatrix}$



$\det A = -2$

$\begin{vmatrix} -25 & 36 \\ -18 & 26 \end{vmatrix} = -25 \times 26 + 36 \times 18 = -2$

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$$



Basis $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ standard basis

$\begin{bmatrix} x \\ y \end{bmatrix} = x e_1 + y e_2$

There is a basis $\{u, v\}$ for $\mathbb{R}^2$ such that $Au = -u$, $Av = 2v$

$A^{10} u = AAA \cdots Au$
$A^2 u = AAu = A(-u) = -Au = u$
$A^3 u = AAA\,u = -u$
$\vdots$
$A^{10} u = u$

$A^2 v = AAv = A(2v) = 2Av = 4v$
$A^3 v = 8v$
$A^{10} v = \underbrace{1024}_{2^{10}} v$

u, v are eigenvectors of A
with corresponding eigenvalues $-1, 2$.

Definition If A is an $n\times n$ matrix, and $v \in \mathbb{R}^n$, then v is an eigenvector for A with eigenvalue λ if $Av = \lambda v$.

How do we find eigenvalues and eigenvectors?

If $Av = \lambda v$ then $Av - \lambda v = 0$ i.e. $Av - \lambda I v = 0$ i.e. $(A-\lambda I)v = 0$.

We should assume $v \neq 0$ is a nonzero null vector for $A-\lambda I$. This can only happen if $\det(A-\lambda I) = 0$.

This condition allows us to solve for the corresponding eigenvalue λ. Solve for λ; and for each value λ (each eigenvalue), solve $(A-\lambda I)v = 0$ for the corresponding eigenvector(s) v.

For $A = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix}$, $A - \lambda I = \begin{bmatrix} -25 & 36 \\ -18 & 26 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -25-\lambda & 36 \\ -18 & 26-\lambda \end{bmatrix}$.

$$\begin{vmatrix} -25-\lambda & 36 \\ -18 & 26-\lambda \end{vmatrix} = (-25-\lambda)(26-\lambda) + 36\cdot 18 = \lambda^2 - \lambda - 2 = (\lambda+1)(\lambda-2)$$

The characteristic polynomial has two roots $\lambda_1 = -1$, $\lambda_2 = 2$ (the two eigenvalues).

To find the corresponding eigenvectors v_1, v_2:

First take $\lambda_1 = -1$ and solve $Av_1 = \underset{\lambda_1 = -1}{-}v_1$ i.e. $(A+I)v_1 = 0$. $A+I = \begin{bmatrix} -24 & 36 \\ -18 & 27 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (by inspection)

Or $\begin{bmatrix} -24 & 36 \\ -18 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ -18 & 27 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$ has null space $\mathrm{Span}\left\{\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}\right\}$ with basis $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} : \quad \begin{aligned} x - \tfrac{3}{2}y &= 0 \\ 0 &= 0 \end{aligned}$$

Introduce a parameter t.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2}t \\ t \end{bmatrix} = t\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$$

We can take v_1 to be any nonzero scalar multiple of $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. I'll take $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. So $Av_1 = \lambda_1 v_1 = -v_1$.

For $\lambda_2 = 2$: Solve $Av_2 = \lambda_2 v_2 = 2v_2$ i.e. $(A-2I)v_2 = 0$ where $A - 2I = \begin{bmatrix}-25 & 36\\ -18 & 26\end{bmatrix} - \begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix} = \begin{bmatrix}-27 & 36\\ -18 & 24\end{bmatrix}$

A null vector of $A-2I$: $v_2 = \begin{bmatrix}4/3\\ 1\end{bmatrix}$ or $\begin{bmatrix}4\\ 3\end{bmatrix}$ So $\begin{bmatrix}-27 & 36\\ -18 & 24\end{bmatrix}\begin{bmatrix}4\\ 3\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$ i.e. $Av_2 = \lambda_2 v_2 = 2v_2$.

$v_1 = \begin{bmatrix}3\\ 2\end{bmatrix}$ and $v_2 = \begin{bmatrix}4\\ 3\end{bmatrix}$ is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

We started with $e_1 = \begin{bmatrix}1\\ 0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\ 1\end{bmatrix}$ as the standard basis.

Check: A is similar D $(A = BDB^{-1})$
so $\operatorname{tr} A = \operatorname{tr} D$, $\underbrace{\det A}_{-2} = \underbrace{\det D}_{-2}$.

trace of $A = \operatorname{tr} A = 1$, $\operatorname{tr} D = 1$

To find A^{10}: two approaches.

Let $B = [v_1 \mid v_2] = \begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}$. Then $AB = A\begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix} = \begin{bmatrix}-3 & 8\\ -2 & 6\end{bmatrix} = \begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix} = BD$, $D = \begin{bmatrix}-1 & 0\\ 0 & 2\end{bmatrix}$ (diagonal matrix)

so $A\cancel{BB^{-1}} = BDB^{-1}$ i.e. $A = BDB^{-1}$.

So $A^{10} = (BDB^{-1})(BDB^{-1})\cdots(BDB^{-1}) = BD^{10}B^{-1} = \begin{bmatrix}3 & 4\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1024\end{bmatrix}\begin{bmatrix}3 & -4\\ -2 & 3\end{bmatrix} = \begin{bmatrix}-8183 & 12276\\ -6138 & 9208\end{bmatrix}$

To check: $\det(A^{10}) = (\det A)^{10} = (-2)^{10} = 1024$. $\qquad \det[\ \downarrow\ ] = 1024$.

$\det A = (-25)(26) - (36)(-18) = -2$.

$\det A = (\det B)(\det D)(\det B^{-1}) = 1 \times (-2) \times 1 = -2$

Second approach: $A^{10} v_1 = v_1$, $A^{10} v_2 = 1024 v_2$

$$\left.\begin{aligned} v_1 &= \begin{bmatrix}3\\ 2\end{bmatrix} = 3e_1 + 2e_2\\ v_2 &= \begin{bmatrix}4\\ 3\end{bmatrix} = 4e_1 + 3e_2 \end{aligned}\right\} \Rightarrow \begin{aligned} e_1 &= 3v_1 - 2v_2 = 3\begin{bmatrix}3\\ 2\end{bmatrix} - 2\begin{bmatrix}4\\ 3\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}\\ e_2 &= -4v_1 + 3v_2 = -4\begin{bmatrix}3\\ 2\end{bmatrix} + 3\begin{bmatrix}4\\ 3\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} \end{aligned}$$

$$A^{10}e_1 = A^{10}(3v_1 - 2v_2) = 3\cdot v_1 - 2\times 1024 v_2 = 3\begin{bmatrix}3\\ 2\end{bmatrix} - 2048\begin{bmatrix}4\\ 3\end{bmatrix} = \begin{bmatrix}-8183\\ -6138\end{bmatrix}$$

$$A^{10}e_2 = A^{10}(-4v_1 + 3v_2) = -4v_1 + 3\times 1024 v_2 = -4\begin{bmatrix}3\\ 2\end{bmatrix} + 3072\begin{bmatrix}4\\ 3\end{bmatrix} = \begin{bmatrix}12276\\ 9208\end{bmatrix}$$

$$A^{10} = \begin{bmatrix}-8183 & 12276\\ -6138 & 9208\end{bmatrix}$$

A and D are similar: they represent the same linear transformation with respect to different choices of basis.

Eg. diagonalize the matrix $A = \begin{bmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$.

$\det A = \begin{vmatrix} 4 & -1 & 1 \\ 2 & 1 & 2 \\ 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 2 & 1 \end{vmatrix} \cdot 3 = 6 \cdot 3 = 18$

First compute the characteristic polynomial $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -1 & 1 \\ 2 & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} (3-\lambda) = [(4-\lambda)(1-\lambda)+2](3-\lambda)$

$= [\lambda^2 - 5\lambda + 6](3-\lambda) = (\lambda-2)(\lambda-3)(3-\lambda) = -(\lambda-2)(\lambda-3)^2$ has roots $2, 3, 3$ (the eigenvalues of A).

Find eigenvector v_1 for $\lambda_1 = 2$: solve $(A - \lambda_1 I)v_1 = 0$ i.e. $\begin{bmatrix} 2 & -1 & 1 \\ 2 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v_1} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $v_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow Av_1 = 2v_1$.

Find eigenvectors v_2, v_3 for $\lambda_2 = \lambda_3 = 3$: solve $(A - 3I)v = 0$ i.e. $\begin{bmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Take $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Note: We want two linearly independent solutions.

Form the matrix $B = [v_1 | v_2 | v_3] = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ whose columns are the eigenvectors. (v_1, v_2, v_3 is our basis of eigenvectors)

Then $AB = BD$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

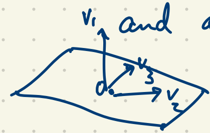
i.e. $A\cancel{B}\cancel{B^{-1}} = BDB^{-1}$

i.e. $\boxed{A = BDB^{-1}}$. We have diagonalized A.

$AB = A[v_1 | v_2 | v_3] = [Av_1 | Av_2 | Av_3] = [2v_1 | 3v_2 | 3v_3] = [v_1 | v_2 | v_3] \begin{bmatrix} 2 & & \\ & 3 & \\ & & 3 \end{bmatrix} = BD$

Check: $\operatorname{tr} A \stackrel{?}{=} \operatorname{tr} D$, $\det A \stackrel{?}{=} \det D$
$8 = 8$, $18 = 18$

$\mathbb{R}^3$ has an eigenvector v_1 with eigenvalue $\lambda_1 = 2$ and an eigenspace $\operatorname{Span}\{v_2, v_3\}$ with eigenvalue 3.



$x - y + z = 0$ ($\operatorname{Span}\{v_2, v_3\}$)

The eigenspace for λ is $\mathrm{Nul}(A-\lambda I) = \{\text{all eigenvectors having eigenvalue } \lambda\}$
$= \{\text{all } v \text{ satisfying } Av = \lambda v\}$.

$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ has a single eigenspace $\mathbb{R}^3$ with eigenvalue 5.

Actually, we don't necessarily have a basis of eigenvectors.

Consider $A = \begin{bmatrix} -7 & 16 \\ -4 & 9 \end{bmatrix}$.

Find the characteristic polynomial $\det(A-\lambda I) = \begin{vmatrix} -7-\lambda & 16 \\ -4 & 9-\lambda \end{vmatrix} = (-7-\lambda)(9-\lambda) + 64 = \lambda^2 - \underbrace{2}_{\mathrm{Tr}\,A=2}\lambda + \underbrace{1}_{\det A = 1} = (\lambda-1)^2$

which has roots 1, 1. (Only one distinct eigenvalue)

Look for eigenvectors: $(A-I)v = 0$ i.e. $\begin{bmatrix} -8 & 16 \\ -4 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Take $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Try to complete this to a basis $v_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $B = [v_1 | v_2] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

$AB = A[v_1 | v_2] = [Av_1 | Av_2] = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix} = B\underbrace{\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}}_{M}$

$AB = BM \iff A = BMB^{-1}$

A, M are similar matrices having the same trace, determinant, characteristic poly.

$Av_2 = \begin{bmatrix} -7 & 16 \\ -4 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$

$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}$

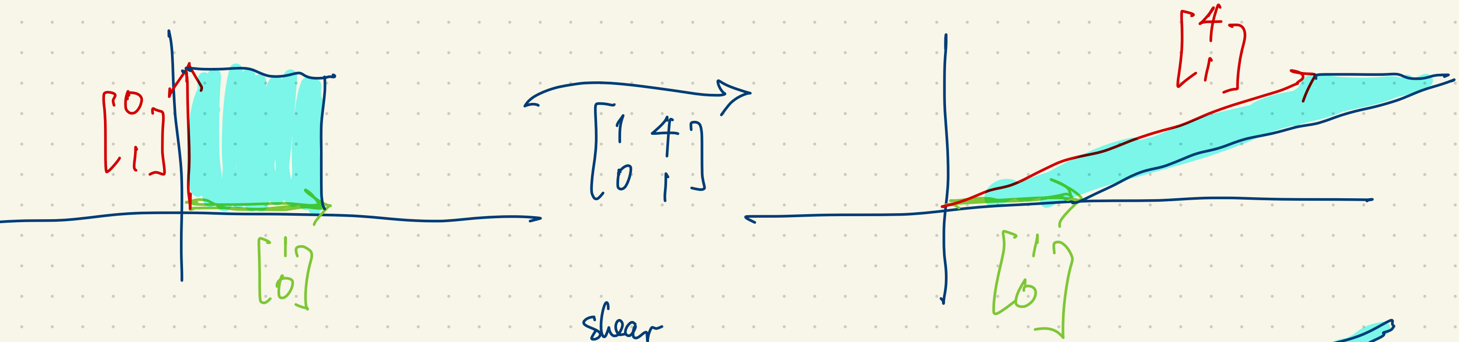
$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{B}\underbrace{\begin{bmatrix} 1 & ? \\ 0 & 1 \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix}}_{AB}$

$B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

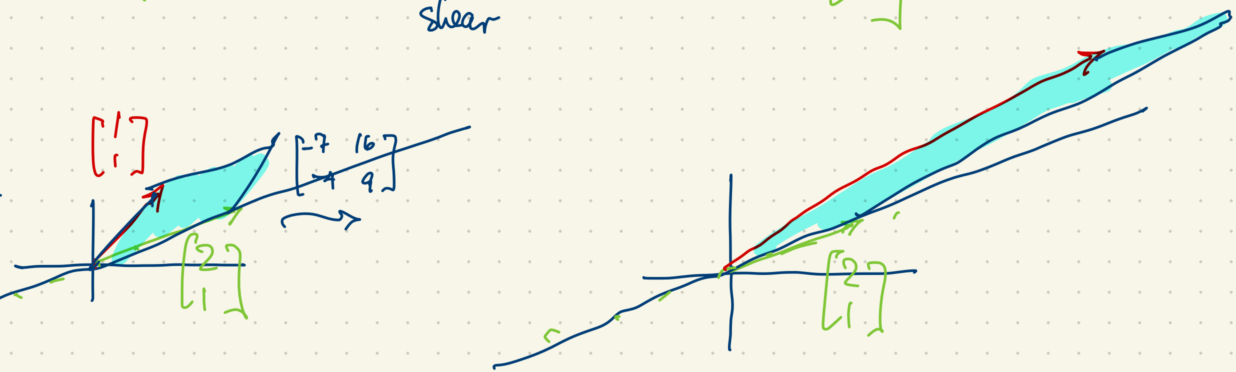
$B^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$M = B^{-1}AB = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 9 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

M :

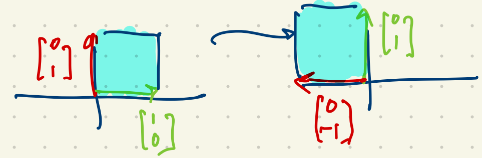


A: also a shear



A is not diagonalizable ;
$\mathbb{R}^2$ does not have a basis
consisting of eigenvectors for A.
(we have one eigenvector only).

An example with no eigenvectors or eigenvalues:



$A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ represents a $90°$ rotation counterclockwise

Algebraically: compute the characteristic polynomial

$$\det(A-\lambda I) = \det\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right) = \begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix} = \lambda^2+1$$

Over $\mathbb{R}$ there are no roots of λ^2+1 (you cannot factor this).

Over $\mathbb{C} = \{a+bi : a,b\in\mathbb{R}\}$, however, we factor $\lambda^2+1 = (\lambda+i)(\lambda-i)$

so the roots $i, -i$ give two eigenvalues in $\mathbb{C}$. $\quad i^2=-1$

Find eigenvectors for A.

$Av_1 = iv_1 \iff (A-iI)v=0$ i.e. $\begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$. Take $v_1 = \begin{bmatrix}i\\ 1\end{bmatrix}$ as an eigenvector.

$Av_2 = -iv_2$, $\quad v_2 = \begin{bmatrix}-i\\ 1\end{bmatrix}$

$A\,[v_1 \mid v_2] = [Av_1 \mid Av_2] = [v_1/v_2]\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$

So $A = BDB^{-1}$, $\quad D = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$, $\quad B = \begin{bmatrix}i & -i\\ 1 & 1\end{bmatrix}$, columns v_1, v_2

$D = B^{-1}AB$

$AB = BD$

$A = BDB^{-1}$

$\{v_1, v_2\}$ is a basis of $\mathbb{C}^2 = \left\{\begin{bmatrix}z_1\\ z_2\end{bmatrix} : z_1, z_2 \in \mathbb{C}\right\}$ is a 2-dimensional vector space over the field $\mathbb{C}$ of complex numbers

$A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$ is not diagonalizable over the real numbers $\mathbb{R}$

but it is diagonalizable over $\mathbb{C}$.

Vector Spaces: Chapter 4

Scalars: real numbers / complex numbers / rational numbers / general fields

A field is a set of scalars in which we can add, subtract, multiply and divide.

A <u>vector space</u> is a set V whose elements are called vectors, including a zero vector $\underline{0}$, and operations $+$, $-$, scalar multiplication satisfying

1. For $\underline{u}, \underline{v} \in V$, $\underline{u}+\underline{v} \in V$. (vector + vector = vector) (scalar + scalar = scalar, ~~scalar + vector~~)
2. $\underline{u}+\underline{v} = \underline{v}+\underline{u}$
3. $(\underline{u}+\underline{v})+\underline{w} = \underline{u}+(\underline{v}+\underline{w})$
4. $\underline{u}+\underline{0} = \underline{u} = \underline{0}+\underline{u}$

For all $\underline{u}, \underline{v}, \underline{w} \in V$ (2–4)

~~vector × vector~~

5. For each $\underline{u} \in V$, there is a vector $-\underline{u} \in V$ such that $\underline{u} + (-\underline{u}) = \underline{0}$
6. Scalar multiplication: For every scalar c and $\underline{u} \in V$, $c\underline{u} \in V$ (scalar × vector = vector)
7. Distributivity: $c(\underline{u}+\underline{v}) = c\underline{u} + c\underline{v}$
8. ” $(c+d)\underline{v} = c\underline{v} + d\underline{v}$
9. Associativity: $(cd)\underline{v} = c(d\underline{v})$
10. $1\underline{u} = \underline{u}$

$0\underline{u} = \underline{0}$ as follows from the axioms: $0\underline{u} + 0\underline{u} = (0+0)\underline{u} = 0\underline{u}$. Add $-0\underline{u}$ to both sides:

scalar (0), vector ($\underline{u}$), vector ($\underline{0}$)

$$(0\underline{u}+0\underline{u})+(-0\underline{u}) = 0\underline{u}+(-0\underline{u}) = \underline{0}$$

By (3), $0\underline{u} + (0\underline{u} + (-0\underline{u})) = \underline{0}$

By (5) $0\underline{u} + \underline{0} = \underline{0}$

$$0\underline{u} = \underline{0}$$

Examples of vector spaces:

$\mathbb{R}^n$ (actually, $\mathbb{R}^{n\times 1}$ is column vectors of length n; $\mathbb{R}^{1\times n}$ is row vectors of length n).

Subspaces of $\mathbb{R}^n$

The set of all polynomials of degree $< n$ in x is an n-dimensional vector space

$$V = \{a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1} : a_0, a_1, a_2, \ldots, a_{n-1} \text{ are scalars}\}.$$

$\{1, x, x^2, \ldots, x^{n-1}\}$ is a basis for V. x is an indeterminate (i.e. not a number, just a symbol).

$\{1, x, x(x-1), x(x-1)(x-2), \ldots, x(x-1)(x-2)\cdots(x-n+1)\}$ is also a basis.

The set of all polynomials in x is a vector space which is infinite-dimensional.

A basis is $\{1, x, x^2, x^3, x^4, \ldots\}$

Examples of polynomials: $5-3x+2x^2$, $1-x^3+3x^7+11x^8$, ...

Not polynomials: $\sin x$, $\sqrt{1+x}$, $x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \cdots$

The set of all functions $\mathbb{R} \to \mathbb{R}$.

As a subspace of this, the continuous functions $\mathbb{R} \to \mathbb{R}$.

An even smaller subspace: differentiable functions $\mathbb{R} \to \mathbb{R}$.

Even smaller: the space of "smooth functions" $V = \{f : \mathbb{R} \to \mathbb{R} : f^{(n)} \text{ exists for all } n \geq 0\}$ (nth derivative)

A linear transformation $T : V \to V$ is defined by $T = D^2 + I$ $\left(D = \frac{d}{dx}\right)$ i.e. $Tf = f'' + f$.

The rank of T is infinite dimensional. T is not one-to-one.

A basis for $\text{Nul}\, T = \{f : Tf = 0\}$ is $\{\sin x, \cos x\}$. $Tf = 0$ iff $f(x) = a\sin x + b\cos x$ for some $a, b \in \mathbb{R}$.

$D : V \to V$ has $\text{Nul}\, D = \{\text{constant functions}\}$ having basis $\{1\}$; $\text{Nul}\, D$ is one-dimensional.

D has eigenvectors! eg. $De^{3x} = 3e^{3x}$. For every $\lambda \in \mathbb{R}$, the set of eigenvectors having eigenvalue λ is one-dimensional with basis $\{e^{\lambda x}\}$.

Fibonacci Numbers

$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21\ 34,\ 55,\ 89,\ 144,\ \ldots$

$F_0 = 0$
$F_1 = 1$
$F_2 = 1$
$F_3 = 2$
$F_4 = 3$ etc.
$F_5 = 8$

Recursive formula $F_n = \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n=1; \\ F_{n-1}+F_{n-2}, & \text{if } n \geqslant 2. \end{cases}$

Consider $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}5\\3\end{bmatrix}, \ldots$

$\quad\quad\quad\ \ v_0 \quad v_1 \quad v_2 \quad v_3 \quad v_4$

So $v_n = \begin{bmatrix}F_{n+1}\\F_n\end{bmatrix}$ so $A = \begin{bmatrix}1&1\\1&0\end{bmatrix}$ defines a map $v_n \longmapsto Av_n = v_{n+1}$ i.e. $A\begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = \begin{bmatrix}F_{n+2}\\F_{n+1}\end{bmatrix} = v_{n+1}$.

Starting with $v_0 = \begin{bmatrix}1\\0\end{bmatrix}$, we get $v_1 = Av_0$, $v_2 = Av_1 = A^2v_0, \ldots, v_n = \begin{bmatrix}F_{n+1}\\F_n\end{bmatrix} = A^n\begin{bmatrix}1\\0\end{bmatrix}$ = first column of A^n.

$A^2 = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$, $A^3 = \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}3&2\\2&1\end{bmatrix}$, $A^4 = \begin{bmatrix}3&2\\2&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}5&3\\3&2\end{bmatrix}, \ldots$

To find an explicit formula for A^n (and thereby F_n), diagonalize A.

Characteristic polynomial of A:

$\det(A - xI) = \det\left(\begin{bmatrix}1&1\\1&0\end{bmatrix} - \begin{bmatrix}x&0\\0&x\end{bmatrix}\right) = \begin{vmatrix}1-x&1\\1&-x\end{vmatrix} = (1-x)(-x) - 1 = x^2 - x - 1 = (x-\alpha)(x-\beta)$ where $\underbrace{\alpha = \frac{1+\sqrt5}{2}}_{\text{golden ratio}}$, $\underbrace{\beta = \frac{1-\sqrt5}{2}}_{-0.618\ldots}$

$\approx 1.618\ldots$

Eigenvector for α: solution of $Av = \alpha v$ i.e. $(A - \alpha I)v = 0$

$\begin{bmatrix}1-\alpha&1\\1&-\alpha\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0$. A nonzero solution is $\begin{bmatrix}\alpha\\1\end{bmatrix}$. Check: $\begin{bmatrix}1-\alpha&1\\1&-\alpha\end{bmatrix}\begin{bmatrix}\alpha\\1\end{bmatrix} = \begin{bmatrix}(1-\alpha)\alpha+1\\\alpha-\alpha\end{bmatrix} = \begin{bmatrix}1+\alpha-\alpha^2\\0\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

Eigenvector for β: $Av = \beta v$ i.e. $(A - \beta I)v = 0$. Take $\begin{bmatrix}\beta\\1\end{bmatrix}$.

$B = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}$ has the eigenvectors as its columns. $AB = \left[A\begin{bmatrix}\alpha\\1\end{bmatrix} \middle| A\begin{bmatrix}\beta\\1\end{bmatrix}\right] = \begin{bmatrix}\alpha^2&\beta^2\\\alpha&\beta\end{bmatrix} = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}\begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix} = BD$, $D = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$.

Diagonalizing A gives $D = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}$. $\quad A\cancel{BB^{-1}} = BDB^{-1}$ i.e. $\boxed{A = BDB^{-1}}$ $\quad D^n = \begin{bmatrix}\alpha&0\\0&\beta\end{bmatrix}^n = \begin{bmatrix}\alpha^n&0\\0&\beta^n\end{bmatrix}$

$A^n = \underbrace{(BD\cancel{B^{-1}})(\cancel{B}D\cancel{B^{-1}})(\cancel{B}D\cancel{B^{-1}})\cdots(\cancel{B}DB^{-1})}_{n\ \text{times}} = BD^nB^{-1}$

$B = \begin{bmatrix}\alpha&\beta\\1&1\end{bmatrix}$ $\quad B^{-1} = \frac{1}{\sqrt5}\begin{bmatrix}1&-\beta\\-1&\alpha\end{bmatrix}$

$\det B = \alpha - \beta = \sqrt5$

$$A^n = BDB^{-1} = \begin{bmatrix} \alpha & \beta \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} 1 & -\beta \\ 1 & \alpha \end{bmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} & \boxed{\alpha\beta^{n+1} - \alpha^{n+1}\beta} \leftarrow \alpha^n - \beta^n \\ \alpha^n - \beta^n & \boxed{\alpha\beta^n - \alpha^n\beta} \leftarrow \alpha^{n-1} - \beta^{n-1} \end{bmatrix}$$

$$\alpha\beta = -1 \qquad \alpha + \beta = 1 \qquad \alpha - \beta = \sqrt{5}$$

$$\alpha\beta = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{1-\sqrt{5}}{2}\right) = \frac{1-5}{4} = \frac{-4}{4} = -1$$

$$v_n = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^{n+1} - \beta^{n+1} \\ \alpha^n - \beta^n \end{bmatrix}$$

so $\boxed{F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}} = \dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$ grows exponentially

(faster than power law n^k)

eg. $F_0 = \dfrac{\alpha^0 - \beta^0}{\sqrt{5}} = \dfrac{1-1}{\sqrt{5}} = 0$

$F_1 = \dfrac{\alpha - \beta}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

$F_2 = \dfrac{\alpha^2 - \beta^2}{\sqrt{5}} = \dfrac{(\alpha+1) - (\beta+1)}{\sqrt{5}} = \dfrac{\sqrt{5}}{\sqrt{5}} = 1$

$F_3 = 2$ etc.

$F_{30} = \dfrac{\alpha^{30} - \beta^{30}}{\sqrt{5}} = 832040$

A 2-dimensional vector space: the solutions of $y'' + y = 0$.

Over $\mathbb{R}$, $\{\sin x, \cos x\}$ is a basis for the solutions:

Over $\mathbb{C}$, $\{e^{ix}, e^{-ix}\}$ is another basis.

If $y = e^{ix}$ then $y' = ie^{ix}$, $y'' = -e^{ix}$, $y'' + y = -e^{ix} + e^{ix} = 0$

Let V be the vector space consisting of all solutions of $y'' + y = 0$.

$D: V \to V$, $Dy = y'$ is a linear transformation.

D is represented by the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ with respect to the first choice of basis:

$D(\underline{a \sin x + b \cos x}) = -b \sin x + a \cos x$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$ Over $\mathbb{R}$, D has no (nonzero) eigenvectors.

But over $\mathbb{C}$, e^{ix} is an eigenvector with eigenvalue i;

e^{-ix} - - - - - - - - - $-i$.

$\{e^{ix}, e^{-ix}\}$ is a basis of V consisting of eigenvectors of D.

Over $\mathbb{R}$: consider the vector space V consisting of all polynomials in x of degree $< n$.

$$V = \{ a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} \; : \; a_0, a_1, \cdots, a_n \in \mathbb{R} \}.$$

$D : V \to V$, $\quad Df(x) = f'(x)$ is linear since $D(af + bg) = (af+bg)' = af' + bg'$
$= aDf + bDg$.

In matrix terminology

$$D(a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + (n-1) a_{n-1} x^{n-2}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 3 & \cdots & 0 \\ \vdots & 0 & 0 & 0 & \cdots & n-1 \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\substack{\text{Not invertible;} \\ \text{it has rank } n-1}} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ \vdots \\ (n-1)a_{n-1} \\ 0 \end{bmatrix}$$

The characteristic polynomial of D is $\det \begin{bmatrix} D - \lambda I \end{bmatrix} = \begin{vmatrix} -\lambda & 1 & & & \\ & -\lambda & 2 & & \bigcirc \\ & & -\lambda & 3 & \\ & \bigcirc & & \ddots & n-1 \\ & & & & -\lambda \end{vmatrix} = (-\lambda)^n$

The only root is $\lambda = 0$. An eigenvector for this eigenvalue is 1. $\quad D1 = 0 = 0 \cdot 1$.

(Eigenvectors for eigenvalue 0 are the same thing as null vectors.)

If we move beyond polynomials then $D = \frac{d}{dx}$ has an eigenvector for every scalar λ : $De^{\lambda x} = \lambda e^{\lambda x}$. So $e^{\lambda x}$ is an eigenvector with eigenvalue λ. This works over both $\mathbb{R}$ and $\mathbb{C}$.
($e^{\lambda x}$ is an "eigenfunction").

Eg. Let V be the set of all rational functions in x of the form $\dfrac{ax+b}{x^2+8x+15}$. First decompose $\dfrac{ax+b}{x^2+8x+15} = \dfrac{ax+b}{(x+3)(x+5)} = \dfrac{A}{x+3} + \dfrac{B}{x+5}$.

We know there exist A, B (for every choice of a, b).

V is a vector space over $\mathbb{R}$.

$$\frac{ax+b}{x^2+8x+15} + \frac{cx+d}{x^2+8x+15} = \frac{(a+c)x+(b+d)}{x^2+8x+15} \in V$$

$$c\,\frac{ax+b}{x^2+8x+15} = \frac{(ca)x+cb}{x^2+8x+15} \in V$$

$\dim V = 2$ because $\dfrac{ax+b}{x^2+8x+15} = a\cdot\dfrac{x}{x^2+8x+15} + b\cdot\dfrac{1}{x^2+8x+15}$ expresses your vector uniquely as a linear combination of $\dfrac{x}{x^2+8x+15}$, $\dfrac{1}{x^2+8x+15}$.

We want to conclude that $\left\{ \frac{1}{x+3}, \frac{1}{x+5} \right\}$ is also a basis.

First note that $\frac{1}{x+3} = \frac{x+5}{(x+3)(x+5)} \in V$ and $\frac{1}{x+5} = \frac{x+3}{(x+3)(x+5)} \in V$.

Eg. Decompose $\frac{7x+11}{x^2+8x+5}$ into its parts by the method of partial fractions.

$$\frac{7x+11}{x^2+8x+5} = \frac{7x+11}{(x+3)(x+5)} = \frac{A}{x+3} + \frac{B}{x+5} = \frac{-5}{x+3} + \frac{12}{x+5}.$$

$$7x+11 = (x+5)A + (x+3)B$$

For $x=-3$, $-10 = 2A$ so $A = -5$.

For $x=-5$, $-24 = -2B$ so $B = 12$

Find eigenvalues and eigenvectors of $A = \begin{bmatrix} 47 & -30 \\ 75 & -48 \end{bmatrix}$.

The characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 47-\lambda & -30 \\ 75 & -48-\lambda \end{vmatrix} = (47-\lambda)(-48-\lambda) + 2250 = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3)$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -3$.

For $\lambda_1 = 2$, v_1 is a null vector of $A - 2I = \begin{bmatrix} 45 & -30 \\ 75 & -50 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 \\ 0 & 0 \end{bmatrix}$ so $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Check: $A v_1 = \begin{bmatrix} 47 & -30 \\ 75 & -48 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ ✓

$= 2 v_1$

For $\lambda_2 = -3$, v_2 is a null vector of $A + 3I = \begin{bmatrix} 50 & -30 \\ 75 & -45 \end{bmatrix} \sim \begin{bmatrix} 5 & -3 \\ 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 5 & -3 \\ 0 & 0 \end{bmatrix}$ so $v_2 = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$

Check: $A v_2 = \begin{bmatrix} 47 & -30 \\ 75 & -48 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} -9 \\ -15 \end{bmatrix} = -3 \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ ✓

Another check: $\operatorname{tr} A = -1$, $\det A = -6$.

A is similar to $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} = D$, $\operatorname{tr} D = -1$, $\det D = -6$. ✓