University of Wyoming Math 2250—Fall 2023 Department of Mathematics

Elementary Linear Algebra

Solutions to the Sample Test

November, 2023

1. From the matrix

$$A = \begin{bmatrix} 0 & \textcircled{1} & 8 & 0 & 3 & 0 & -2 & 0 \\ 0 & 0 & 0 & \textcircled{1} & -7 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{1} & 9 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{1} \end{bmatrix}.$$

we observe that x_1, x_3, x_5, x_7 are free variables while x_2, x_4, x_6, x_8 are basic. Introducing four parameters a_1, a_2, a_3, a_4, the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} = \begin{bmatrix} a_1 \\ -8a_2-3a_3+2a_4 \\ a_2 \\ 7a_3-a_4 \\ a_3 \\ -9a_4 \\ a_4 \\ 0 \end{bmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4$$

where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -8 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -3 \\ 0 \\ 7 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ -9 \\ 1 \\ 0 \end{bmatrix}.$$

Now the four vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_2, \mathbf{v}_4$ are linearly indepedent and they span $\operatorname{Nul} A$ (so they form a *basis* for $\operatorname{Nul} A$).

2. Since A has rank 2, and its first two columns are linearly independent, these two columns (call them $\mathbf{u}_1$ and $\mathbf{u}_2$) must form a basis for the column space. By inspection, or by solving the appropriate linear systems, we see that the third and fourth columns must be $\mathbf{u}_2-2\mathbf{u}_1$ and $\mathbf{u}_1+2\mathbf{u}_2$ respectively. We read off the entries $a = -5$ and $b = 14$.

   Second solution: Instead of using columns, you could use the same argument with rows. The first two rows of A (let's call them $\mathbf{v}_1$ and $\mathbf{v}_2$) must be a basis for the row space. Then rows three and four must be $4\mathbf{v}_1-5\mathbf{v}_2$ and $\frac{7}{2}\mathbf{v}_2-3\mathbf{v}_1$, respectively. Once again, this gives $a = -5$ and $b = 14$.

3. Since

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

and

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}2\\3\\4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

we have the images of all three standard basis vectors under T. These form the columns of the standard matrix of T, which is therefore

$$\begin{bmatrix}2&0&0\\3&0&0\\4&0&0\end{bmatrix}.$$

4. (a) Yes; $U \cap V$ is a subspace. Since $\mathbf{0} \in U$ and $\mathbf{0} \in V$, we have $\mathbf{0} \in U \cap V$. If $a, b \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in U \cap V$, then both U and V contain $\mathbf{x}$ and $\mathbf{y}$, so $a\mathbf{x} + b\mathbf{y} \in U$ and $a\mathbf{x} + b\mathbf{y} \in V$, so $a\mathbf{x} + b\mathbf{y} \in U \cap V$.
   (b) No; $U \cup V$ is not a subspace in general. For example if $n = 3$ and $U = \text{Span}\{\mathbf{e}_1, \mathbf{e}_2\}$ (the xy-plane) and $V = \text{Span}\{\mathbf{e}_1, \mathbf{e}_3\}$ (the xz-plane) then $U \cup V$ is not a subspace; we have $(1,1,0), (1,0,1) \in U \cup V$ but $(1,1,0) + (1,0,1) = (2,1,1) \notin U \cup V$.
   (c) Yes; $U+V$ is a subspace. Since $\mathbf{0} \in U$ and $\mathbf{0} \in V$, we have $\mathbf{0} = \mathbf{0}+\mathbf{0} \in U+V$. Now let $a, b \in \mathbb{R}$ and take any two vectors $\mathbf{u}+\mathbf{v} \in U+V$ and $\mathbf{u}'+\mathbf{v}' \in V$, where $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{v}, \mathbf{v}' \in V$; then $a(\mathbf{u}+\mathbf{v})+b(\mathbf{u}'+\mathbf{v}') = (a\mathbf{u}+b\mathbf{u}')+(a\mathbf{v}+b\mathbf{v}') \in U+V$. Alternatively, it is not hard to see from the definition that $U+V = \text{Span}(U \cup V)$ which is a subspace of $\mathbb{R}^n$ (since the span of any set of vectors is a subspace).

5. (a) F (b) F (c) T (d) T (e) T (f) F (g) T (h) F (i) F (j) F

Comments (not required, but provided here for your benefit):

(a) As a counterexample, take $\mathbf{u} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_2$, $\mathbf{w} = \mathbf{e}_1 + \mathbf{e}_2$ where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard basis of $\mathbb{R}^n$.
(b) Take the same counterexample as in (a), with $n = 3$.
(c) If $\mathbf{v} \in \text{Nul}\, T$, then $T(\mathbf{v}) = \mathbf{0}$ so $ST(\mathbf{v}) = S(\mathbf{0}) = \mathbf{0}$, i.e. $\mathbf{v} \in \text{Nul}(ST)$.
(d) Suppose $A\mathbf{v}_i = \mathbf{0}$ for $i = 1, 2, \ldots, k$, and $\mathbf{v} = a\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k$ where $a_1, a_2, \ldots, a_k \in \mathbb{R}$. Then

$$A\mathbf{v} = a_1 A\mathbf{v}_1 + a_2 A\mathbf{v}_2 + \cdots + a_k A\mathbf{v}_k = a_1\mathbf{0} + a_2\mathbf{0} + \cdots + a_k\mathbf{0} = \mathbf{0}.$$

(e) As explained in class, $A = BM$ where B is invertible $m \times m$. If $A\mathbf{v} = \mathbf{0}$ then $M\mathbf{v} = BA\mathbf{v} = B\mathbf{0} = \mathbf{0}$. Conversely, if $M\mathbf{v} = \mathbf{0}$ then $A\mathbf{v} = B^{-1}M\mathbf{v} = B^{-1}\mathbf{0} = \mathbf{0}$.
(f) As in (e), we have $A = BM$ for some invertible $m \times m$ matrix B. If $M\mathbf{v} = \mathbf{b}$, then $A\mathbf{v} = BM\mathbf{v} = B\mathbf{b}$, which is not the same as $\mathbf{b}$.
(g) Neither row is a scalar multiple of the other.
(h) The columns satisfy the nontrivial relation $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right] - 2\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right] + \left[\begin{smallmatrix}5\\6\end{smallmatrix}\right] = \mathbf{0}$.

(i) As a counterexample, consider a list of linearly dependent vectors $\mathbf{v}_1 = \mathbf{e}_1$ and $\mathbf{v}_2 = \cdots = \mathbf{v}_k = \mathbf{0}$, which however yield the nonzero vector $\mathbf{e}_1$ among its linear combinations. (Here I mean $\mathbf{e}_1$ to be the first standard basis vector, but any nonzero vector would serve in its place.)

(j) As a counterexample, take $\mathbf{v}_1, \ldots, \mathbf{v}_k$ to be a basis (such as the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$) together with additional vectors thrown in. In this case we have $k > n$.