University of Wyoming — Department of Mathematics

Math 2250—Fall 2023

Elementary Linear Algebra

Sample Test

November, 2023

This sample test is intended to resemble the test (Wednesday, November 8, 2023 during class time) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class prior to the test (excluding determinants). This includes Sections 1.1–1.5, 1.7–1.9, 2.1–2.3, 2.8–2.9 of the textbook.

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 50 minutes. Total value of questions: 100 points (plus 10 bonus points).

1. *(20 points)* Consider the matrix

$$A=\begin{bmatrix}0&1&8&0&3&0&-2&0\\0&0&0&1&-7&0&1&0\\0&0&0&0&0&1&9&0\\0&0&0&0&0&0&0&1\end{bmatrix}.$$

Note that A is in reduced row echelon form. Determine the dimension of $\operatorname{Nul} V$, and write down a basis for this null space.

2. *(20 points)* Given that the following matrix has rank 2, determine the constants a and b:

$$\begin{bmatrix}1&3&1&7\\2&4&0&10\\3&1&a&5\\4&5&-3&b\end{bmatrix}$$

3. *(20 points)* Write down the standard matrix of the linear transformation $T:\mathbb{R}^3\to\mathbb{R}^3$ satisfying

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\3\\4\end{bmatrix}.$$

4. *(20 points)* Let U and V be subspaces of $\mathbb{R}^n$.
   (a) Is the intersection $U\cap V$ a subspace of $\mathbb{R}^n$? Explain. (Recall that $U\cap V$ is the set of all vectors $\mathbf{x}$ such that $\mathbf{x}\in U$ *and* $\mathbf{x}\in V$.)
   (b) Is the union $U\cup V$ a subspace of $\mathbb{R}^n$? Explain. (Recall that $U\cup V$ is the set of all vectors $\mathbf{x}$ in either U *or* in V, possibly in both.)
   (c) Define $U+V$ to be the set of all vectors of the form $\mathbf{u}+\mathbf{v}$ where $\mathbf{u}\in U$ and $\mathbf{v}\in V$. Is $U+V$ a subspace of $\mathbb{R}^n$? Explain.

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three vectors in $\mathbb{R}^n$, none of which is a scalar multiple of the other two, then the three vectors must be linearly independent. ______*(True/False)*

(b) If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ are three vectors, none of which is a scalar multiple of the other two, then the three vectors must span $\mathbb{R}^3$. ______*(True/False)*

(c) If $T : \mathbb{R}^\ell \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^n$, then $\mathrm{Nul}(T)$ is a subset of $\mathrm{Nul}(ST)$. ______*(True/False)*

(d) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are solutions of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$, then every linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is a solution of the same linear system. ______*(True/False)*

(e) If M is an $m \times n$ matrix and A is its reduced row-echelon form, then the linear system $M\mathbf{x} = \mathbf{0}$ has the same solutions as the system $A\mathbf{x} = \mathbf{0}$ (i.e. the two systems are equivalent). ______*(True/False)*

(f) Let M be an $m \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^n$ a column vector. Then the linear system $M\mathbf{x} = \mathbf{b}$ is equivalent to $A\mathbf{x} = \mathbf{b}$ where A is the reduced row-echelon form for M (i.e. any solution of one system also solves the other). ______*(True/False)*

(g) The rows of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ are linearly independent. ______*(True/False)*

(h) The columns of the matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ are linearly independent. ______*(True/False)*

(i) If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ are linearly dependent, then the only linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is the zero vector $\mathbf{0}$. ______*(True/False)*

(j) If $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} = \mathbb{R}^n$, then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ must be linearly independent. ______*(True/False)*