Solutions to Sample Exam

December, 2023

1. The matrix $\left[\begin{smallmatrix}0 & -3\\ 1 & -2\end{smallmatrix}\right]$ works. All that is required is a 2×2 matrix having trace -2 and determinant 3. In general, the matrix $\left[\begin{smallmatrix}0 & -c\\ 1 & -b\end{smallmatrix}\right]$ has characteristic polynomial x^2+bx+c. This answer is very far from unique. You can prescribe one row (or column) to be any vector you want, and solve for the remaining two entries so that the characteristic polynomial is as desired. Similarly, every monic polynomial of degree n can be realized as the characteristic polynomial of an $n\times n$ matrix.

2. *First solution*: Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_r\}$ be a basis for U, and $\{\mathbf{v}_1,\ldots,\mathbf{v}_s\}$ be a basis for V. The union of the two bases gives $r+s>n$ vectors in $\mathbb{R}^n$, which must therefore be linearly dependent. This means that there exist scalars $a_1,\ldots,a_r,b_1,\ldots,b_s$, *not all zero*, such that
$$a_1\mathbf{u}_1+a_2\mathbf{u}_2+\cdots+a_r\mathbf{u}_r+b_1\mathbf{v}_1+b_2\mathbf{v}_2+\cdots+b_s\mathbf{v}_s=\mathbf{0}.$$
This gives
$$a_1\mathbf{u}_1+a_2\mathbf{u}_2+\cdots+a_r\mathbf{u}_r \;=\; -b_1\mathbf{v}_1-b_2\mathbf{v}_2-\cdots-b_s\mathbf{v}_s\in U\cap V.$$
(It is clear from the left side that this vector is in U, and from the second expression that it is in V; this is why it must be in $U\cap V$.) If $U\cap V=\{\mathbf{0}\}$ then the vector above is zero; but then $a_1=a_2=\cdots=a_r=0$ (since the $\mathbf{u}_i$'s are linearly independent) and $b_1=b_2=\cdots=b_s=0$ (since the $\mathbf{v}_j$'s are linearly independent). This contradicts the fact that the a_i's and b_j's cannot all be zero.

   Second solution: Let $r=\dim U$ and $s=\dim V$, so that $r+s>n$. Now U is the set of simultaneous solutions of $n-r$ homogeneous linear equations in n unknowns, and V is the set of simultaneous solutions of $n-s$ homogeneous linear equations in n unknowns. Listing all the $(n-r)+(n-s)=2n-r-s$ equations together, the set of simultaneous solutions is $U\cap V$, and this has dimension at least $n-(2n-r-s)=r+s-n\geqslant 1$. So $U\cap V$ contains a nonzero vector.

   That is (and this is not a different proof, just another way of presenting the first argument): $U=\operatorname{Nul}A$ and $V=\operatorname{Nul}B$ where A is $(n-r)\times n$ and V is $(n-s)\times n$. Clearly, $U\cap V=\operatorname{Nul}M$ where the partitioned matrix $M=\left[\begin{smallmatrix}A\\ B\end{smallmatrix}\right]$ is $(2n-r-s)\times n$. Since $\operatorname{rank}M\leqslant 2n-r-s$, $\dim(U\cap V)=\dim\operatorname{Nul}M\geqslant n-(2n-r-s)=r+s-n\geqslant 1$.

3. The indicated parallelepiped is the image of the standard cube $0\leqslant x,y,z\leqslant 1$ under the matrix transformation $\mathbf{x}\mapsto A\mathbf{x}$ where A is the 3×3 matrix whose three columns are the vectors given. Since the standard cube has volume 1, its image (the parallelepiped) has volume equal to (the absolute value of)

$$\det A = \begin{vmatrix} -1 & 3 & 1 \\ 3 & 2 & 4 \\ 1 & 1 & 1 \end{vmatrix} = 6.$$

4. Six is the maximum number of nonzero entries in any 4×4 matrix in reduced row echelon form. This number is realized for matrices of the form

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The number of nonzero entries is maximized when the pivots are in the leftmost columns. For 0, 1 or 4 pivots, this gives at most 0, 4 or 4 nonzero entries respectively.

5. The matrix

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 5 \\ 1 & 5 & 9 \end{bmatrix}$$

works. Here the first two columns were the first two vectors given. It is then easy to fill in the third column in such a way as to achieve the desired null vector.

6. S is linear; T is not. Denote $m(x) = 7x^2 + 3x - 5$; then $Sf(x) = m(x)f(x)$ so

$$S(af(x)+bg(x)) = m(x)(af(x)+bg(x)) = am(x)f(x)+bm(x)g(x) = aSf(x)+bSg(x)$$

for all $f(x), g(x) \in V$ and all $a, b \in \mathbb{R}$. However, $Tx = 7x^2+3x-5$ whereas $T(2x) = 28x^2+6x-5$ is not the same as $2Tx = 14x^2+6x-10$. Worse yet, $T0 = -5 \neq 0$.

7. (a) T (b) T (c) T (d) F (e) T (f) F (g) F (h) T (i) F (j) T

Comments (not required, but provided here for your benefit):

(a) The column space of A^T is essentially the row space of A (but with all the vectors transposed), and these have the same dimension.

(b) An $n \times n$ matrix A has 0 as an eigenvalue iff $A\mathbf{v} = 0\mathbf{v} = \mathbf{0}$ for some nonzero vector $\mathbf{v}$, iff $\mathbf{v} \in \operatorname{Nul} A$.

(c) If A is invertible then its reduced row echelon form is I, whence $A \sim I$. The same is true for B, giving $A \sim I \sim B$. The converse also holds.

(d) Consider $A = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, having 2 as an eigenvalue; and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 3 \end{smallmatrix}\right]$, having 3 as an eigenvalue. Here $AB = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ does *not* have 6 as an eigenvalue.

(e) $A + cI$ is invertible iff $\det(A + cI) \neq 0$. Since the characteristic polynomial of A has at most n roots, there are *at most* n values of c for which $A + cI$ is not invertible.

(f) Take $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to be the standard basis of $U = \mathbb{R}^3$. Also take $\mathcal{B}' = \{\mathbf{e}_1 + \mathbf{e}_2\}$, so that $U' = \operatorname{Span} B'$ is a line through the origin in $\mathbb{R}^3$. Clearly $U \cap U' = U'$ is one-dimensional, but $\mathcal{B} \cap \mathcal{B}' = \varnothing$.

(g) The shear $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ has only one eigenvalue, namely 1.

(h) As pointed out in class, the characteristic polynomial of a 3×3 matrix has degree 3, so this polynomial must have a root.

(i) The shear $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ also has characteristic polynomial $(1 - \lambda)^2$.

(j) Suppose λ is an eigenvalue of an invertible matrix A; and let $\mathbf{v}$ be a corresponding eigenvector, so that $A\mathbf{v} = \lambda\mathbf{v}$. Then $\mathbf{v} = A^{-1}(A\mathbf{v}) = A^{-1}(\lambda\mathbf{v}) = \lambda A^{-1}\mathbf{v}$. Since $\mathbf{v} \neq \mathbf{0}$, we must have $A^{-1}\mathbf{v} = \frac{1}{\lambda}\mathbf{v}$.