University of Wyoming — Math 2250—Fall 2023 — Department of Mathematics

Elementary Linear Algebra

Final Examination (SAMPLE ONLY)

December, 2023

This sample exam is intended to resemble the final examination (10:15 am–12:15 pm, Wednesday, December 13 in our usual classroom, BU 209) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class this semester, with greater emphasis on the later material (general vector spaces; determinants; and eigenvalues and eigenvectors).

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes.

1. *(12 points)* Find a 2×2 matrix whose characteristic polynomial is $x^2 + 2x + 3$.

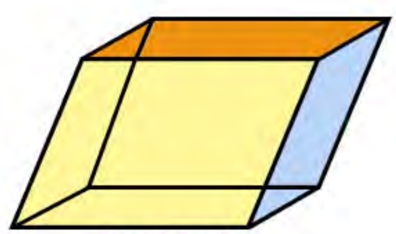
2. *(12 points)* Let U and V be subspaces of $\mathbb{R}^n$. If $\dim U + \dim V > n$, show that there is a nonzero vector $\mathbf{x} \in U \cap V$ (i.e. $\mathbf{x} \in U$ and $\mathbf{x} \in V$).

   Hint: Consider a basis for U and a basis for V.

3. *(12 points)* Find the volume of the parallelepiped having the origin as one of its eight vertices, and the vectors

$$\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

   as three of its edges.



Parallelepiped

4. *(10 points)* What is the maximum possible number of nonzero entries in a 4×4 matrix in reduced row echelon form?

5. *(12 points)* Give an example of a 3×3 matrix having the vectors

$$\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$$

in its column space, and the vector

$$\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

in its null space.

6. *(12 points)* Let $V = \mathbb{R}[x]$, the set of all polynomials in x with real coefficients. This is a vector space (a fact which you may assume). Consider the two transformations $S : V \to V$ and $T : V \to V$ defined by

$$Sf(x) = (7x^2 + 3x - 5)f(x)\,; \qquad Tf(x) = 7f(x)^2 + 3f(x) - 5.$$

Show that one of these transformations is linear and the other is not.

7. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) If A is an $n \times n$ matrix, then its transpose A^T has the same rank as A.
______*(True/False)*

(b) An $n \times n$ matrix is invertible if and only if 0 is *not* one of its eigenvalues.
______*(True/False)*

(c) If A and B are invertible $n \times n$ matrices, then one can obtain B from A by a sequence of elementary row operations. ______*(True/False)*

(d) If A has an eigenvalue λ and B has an eigenvalue μ, then AB has $\lambda\mu$ as an eigenvalue. ______*(True/False)*

(e) If A is an $n \times n$ matrix, then there always exists a constant c such that $A + cI_n$ is invertible. ______*(True/False)*

(f) If U and U' are subspaces of $\mathbb{R}^n$ with bases $\mathcal{B}$ and $\mathcal{B}'$ respectively, then $\mathcal{B} \cap \mathcal{B}'$ is a basis for the subspace $U \cap U'$. ______*(True/False)*

(g) Every $n \times n$ matrix has n distinct eigenvalues. ______*(True/False)*

(h) Every real 3×3 matrix has at least one nonzero eigenvector. ______*(True/False)*

(i) The *only* $n \times n$ matrix with characteristic polynomial $(1 - \lambda)^n$ is the identity matrix I_n. ______*(True/False)*

(j) If λ is an eigenvalue of an invertible $n \times n$ matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. ______*(True/False)*