Solutions to HW3

1. (a) $\begin{vmatrix} a & 2a+b \\ b & a+2b \end{vmatrix} = a(a+2b) - b(2a+b) = a^2 - b^2.$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 9 & 16 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 8 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{vmatrix} = 6.$

Note that this is a Vandermonde determinant with value $(3-1)(4-1)(4-3) = 6$.

(c)
$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & -3 \\ 3 & 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 2 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix} = 2\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{vmatrix}$$
$$= 2\begin{vmatrix} 0 & 0 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} 0 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 0 \end{vmatrix} = -2\begin{vmatrix} 0 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & -2 & -6 \end{vmatrix} = 4\begin{vmatrix} 0 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$
$$= -4\begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = -4\begin{vmatrix} 1 & 0 \\ 1 & 3 \end{vmatrix} = -12.$$

2. (a)
$$\det(A-\lambda I) = \begin{vmatrix} -7-\lambda & -5 & 1 \\ 8 & 7-\lambda & -2 \\ 8 & 5 & -\lambda \end{vmatrix} = \begin{vmatrix} -7-\lambda & -5 & 1 \\ 8 & 7-\lambda & -2 \\ 0 & \lambda-2 & 2-\lambda \end{vmatrix}$$
$$= (\lambda-2)\begin{vmatrix} -7-\lambda & -5 & 1 \\ 8 & 7-\lambda & -2 \\ 0 & 1 & -1 \end{vmatrix} = (\lambda-2)\begin{vmatrix} -7-\lambda & -4 & 1 \\ 8 & 5-\lambda & -2 \\ 0 & 0 & -1 \end{vmatrix}$$
$$= -(\lambda-2)\big[(-7-\lambda)(5-\lambda)+32\big] = -(\lambda-2)(\lambda^2+2\lambda-3) = -(\lambda-2)(\lambda-1)(\lambda+3).$$

(b) The eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = -3$.

(c) We find the eigenvectors $\mathbf{v}_i$ as null vectors of $A - \lambda_i I$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

3. (a) The most obvious basis for U is probably

$$(x-1)(x-3), \quad x(x-1)(x-3), \quad x^2(x-1)(x-3).$$

These three polynomials have degree at most 4, so they are in V; and they vanish at both 1 and 3, so they are in U. Polynomials in U are uniquely expressible in the form $(a+bx+cx^2)(x-1)(x-3)$, which is uniquely expressible as a linear combination of our basis vectors with coefficients a, b, c respectively.

(b) By (a), $\dim U = 3$.

(c) The matrix of T with respect to the standard basis $1, x, x^2, x^3, x^4$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The columns of this matrix represent the images of the five basis vectors

$$\begin{aligned} T(1) &= 1, \\ T(x) &= 1 + x, \\ T(x^2) &= 1 + 2x + x^2, \\ T(x^3) &= 1 + 3x + 3x^2 + x^3, \\ T(x^4) &= 1 + 4x + 6x^2 + 4x^3 + x^4. \end{aligned}$$