Solutions to HW2

1. Recall that in each case the columns of A are given by $T(\mathbf{e}_1)$ and $T(\mathbf{e}_2)$ where $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

   (a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 0.5 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (c) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ (d) $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

2. We first obtain the reduced row echelon form M for A:

$$A = \begin{bmatrix} 1 & 2 & 4 & 4 & -2 & -1 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ -1 & -2 & -1 & 1 & 3 & 0 \\ 2 & 4 & 7 & 4 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 4 & -2 & -1 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 5 & 1 & -1 \\ 2 & 4 & 7 & 4 & -2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 & 4 & -2 & -1 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 5 & 1 & -1 \\ 0 & 0 & -1 & -4 & 2 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -12 & 6 & 19 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 5 & 1 & -1 \\ 0 & 0 & -1 & -4 & 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -12 & 6 & 19 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 3 & 5 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -12 & 6 & 19 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & -4 & 4 & 8 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -12 & 6 & 19 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -12 & 6 & 19 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 0 & -6 & -5 \\ 0 & 0 & 1 & 3 & -1 & -3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 0 & -6 & -5 \\ 0 & 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = M$$

   (a) A basis for Col A is given by $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \\ 4 \end{bmatrix}$. These are columns 1,3,4 of A (the pivot columns).

   (b) The dimension of Col A is 3.

   (c) Nul A = Nul M consists of vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -2r+6s+5t \\ r \\ -2s-3t \\ s+2t \\ s \\ t \end{bmatrix} = r\mathbf{v}_1 + s\mathbf{v}_2 + t\mathbf{v}_3, \quad \text{where } \mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 2 \\ 0 \\ 1 \end{bmatrix},$$

and the parameters $r, s, t \in \mathbb{R}$ are arbitrary scalars. Here $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\operatorname{Nul} A$.

(d) The dimension of $\operatorname{Nul} A$ is 3.

(e) A basis for $\operatorname{Row} A = \operatorname{Row} M$ is formed by the three nonzero rows of M, these being the vectors $\begin{bmatrix} 1 & 2 & 0 & 0 & -6 & -5 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 & 0 & 2 & 3 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 & 1 & -1 & -2 \end{bmatrix}$.

(f) The dimension of $\operatorname{Row} A$ is 3.

3. Here is the short version: Write down the linear equations saying that $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $\operatorname{Nul} A$. These equations tell us simply that each row of A must be in the null space of the matrix whose rows are $\mathbf{v}_1^T$ and $\mathbf{v}_2^T$ (the transoposes of $\mathbf{v}_1$ and $\mathbf{v}_2$). So by finding a basis for this null space, one obtains the two rows of A. To describe this more completely, I will choose to use the language of matrix transposes.

If A and B are matrices of size $\ell \times m$ and $m \times n$ respectively, then the matrix AB is $\ell \times n$. In this case the transposed matrices A^T and B^T have size $m \times \ell$ and $n \times m$ respectively. Thus both the matrices $(AB)^T$ and B^TA^T are of size $n \times \ell$; and indeed, the identity $(AB)^T = B^TA^T$ is easily verified. (The (i, j)-entry of AB is the 'dot product' of the i^{th} row of A with the j^{th} column of B. This coincides with the (j, i)-entry of B^TA^T, this being the 'dot product' of the j^{th} row of B^T with the i^{th} column of A^T.)

Now take $U = \operatorname{Col} B$ where B is the 4×2 matrix with columns $\mathbf{v}_1, \mathbf{v}_2$; and we require a 2×4 matrix A of rank 2 satisfying $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Transposing both sides gives $B^TA^T = (AB)^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Thus the columns of A^T form a basis for $\operatorname{Nul} B^T$. Now

$$B^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 4 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & -4 & -1 \\ 0 & \textcircled{1} & 4 & 2 \end{bmatrix} = M$$

and so $\operatorname{Nul} B^T = \operatorname{Nul} M$ consists of all vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4s+t \\ -4s-2t \\ s \\ t \end{bmatrix} = s\mathbf{w}_1 + t\mathbf{w}_2, \quad \text{where } \mathbf{w}_1 = \begin{bmatrix} 4 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{w}_2 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

and the parameters $s, t \in \mathbb{R}$ are arbitrary. Since $\{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for $\operatorname{Nul} B^T$, we may take $A = \begin{bmatrix} 4 & -4 & 1 & 0 \\ 1 & -2 & 0 & 1 \end{bmatrix}$, the matrix whose rows are the (transposed) vectors $\mathbf{w}_1, \mathbf{w}_2$.

4. The columns of A are the column vectors $T(x^3), T(x^2), T(x), T(1)$; thus

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix}.$$

Alternatively, this is the unique matrix satisfying

$$A\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = T(p) = \begin{bmatrix} d \\ a+b+c+d \\ 8a+4b+2c+d \\ 27a+9b+3c+d \end{bmatrix} \qquad \text{for all } a, b, c, d \in \mathbb{R}.$$

(a)
$$\left[\begin{array}{cccc|cccc} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 8 & 4 & 2 & 1 & 0 & 0 & 1 & 0 \\ 27 & 9 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 8 & 4 & 2 & 1 & 0 & 0 & 1 & 0 \\ 27 & 9 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -4 & -6 & -7 & 0 & -8 & 1 & 0 \\ 27 & 9 & 3 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -4 & -6 & -7 & 0 & -8 & 1 & 0 \\ 0 & -18 & -24 & -26 & 0 & -27 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -4 & -6 & -7 & 0 & -8 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -18 & -24 & -26 & 0 & -27 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{4} & 0 & 2 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -18 & -24 & -26 & 0 & -27 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{4} & 0 & 2 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & \frac{11}{2} & 0 & 9 & -\frac{9}{2} & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{4} & 0 & 2 & -\frac{1}{4} & 0 \\ 0 & 0 & 3 & \frac{11}{2} & 0 & 9 & -\frac{9}{2} & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{4} & 0 & 2 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{11}{6} & 0 & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{7}{4} & 0 & 2 & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{4} & \frac{11}{4} & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & -1 & \frac{7}{2} & 2 & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & \frac{5}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\right] \quad \text{so } A^{-1} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ 1 & -\frac{5}{2} & 2 & -\frac{1}{2} \\ -\frac{11}{6} & 3 & -\frac{3}{2} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & 3 & -3 & 1 \\ 6 & -15 & 12 & -3 \\ -11 & 18 & -9 & 2 \\ 6 & 0 & 0 & 0 \end{bmatrix}.$$

(b) $A^{-1}\begin{bmatrix} 3 \\ -3 \\ -5 \\ 15 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \\ -2 \\ 3 \end{bmatrix}$, so $p(x) = 3x^3 - 7x^2 - 2x + 3$. We checked that this polynomial has the four required values $3, -3, -5, 15$ for $x = 0, 1, 2, 3$.