University of Wyoming — Math 2250—Fall 2023 — Department of Mathematics

Elementary Linear Algebra

SOLUTIONS to Final Examination, December, 2023

1. $AB = BD$ where $B = \left[\begin{smallmatrix}1&1\\2&3\end{smallmatrix}\right]$, $D = \left[\begin{smallmatrix}1&0\\0&2\end{smallmatrix}\right]$ and $B^{-1} = \left[\begin{smallmatrix}3&-1\\-2&1\end{smallmatrix}\right]$, so $A = BDB^{-1} = \left[\begin{smallmatrix}-1&1\\-6&4\end{smallmatrix}\right]$.

2. (a) $A = \begin{bmatrix}1&2&0&-1\\0&1&1&1\\1&0&-1&2\\2&3&0&2\end{bmatrix}$. (The columns are given by the images $T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3), T(\mathbf{e}_4)$ of the four standard basis vectors.)

(b)
$$A = \begin{bmatrix}1&2&0&-1\\0&1&1&1\\1&0&-1&2\\2&3&0&2\end{bmatrix} \sim \begin{bmatrix}1&2&0&-1\\0&1&1&1\\0&-2&-1&3\\2&3&0&2\end{bmatrix} \sim \begin{bmatrix}1&2&0&-1\\0&1&1&1\\0&-2&-1&3\\0&-1&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&-2&-3\\0&1&1&1\\0&-2&-1&3\\0&-1&0&4\end{bmatrix}$$
$$\sim \begin{bmatrix}1&0&-2&-3\\0&1&1&1\\0&0&1&5\\0&-1&0&4\end{bmatrix} \sim \begin{bmatrix}1&0&-2&-3\\0&1&1&1\\0&0&1&5\\0&0&1&5\end{bmatrix} \sim \begin{bmatrix}1&0&-2&-3\\0&1&1&1\\0&0&1&5\\0&0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&0&7\\0&1&1&1\\0&0&1&5\\0&0&0&0\end{bmatrix}$$
$$\sim \begin{bmatrix}1&0&0&7\\0&1&0&-4\\0&0&1&5\\0&0&0&0\end{bmatrix}.$$

Since A has rank 3, $\dim(\mathrm{Nul}\,A) = 1$ and by inspection, $\mathrm{Nul}\,A$ has basis $\left\{\left[\begin{smallmatrix}-7\\4\\-5\\1\end{smallmatrix}\right]\right\}$.

(c) No; $\{T(\mathbf{x}) : \mathbf{x} \in \mathbb{R}^4\} = \mathrm{Col}\,A$ has dimension 3, so it does not equal $\mathbb{R}^4$.

(d) The rank of A is 3 (the number of pivots in the reduced row echelon form above).

(e) No, T is not invertible since its rank (the rank of A) is less than 4. There are many ways to say this: $\det A = 0$; T has a nonzero null space; the column space of A is a proper subspace of $\mathbb{R}^4$.

3. (a) V has basis $\left\{\left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&1&0\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&1\\0&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&0\\1&0&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&0\\0&1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0&0&0\\0&0&1\end{smallmatrix}\right]\right\}$ so $\dim V = 6$.

(b) Using $A^{-1} = \left[\begin{smallmatrix}2&-1\\-5&3\end{smallmatrix}\right]$, we solve for $M = A^{-1}\left[\begin{smallmatrix}0&1&2\\3&4&5\end{smallmatrix}\right] = \left[\begin{smallmatrix}-3&-2&-1\\9&7&5\end{smallmatrix}\right]$.

(c,d) Yes, T is both one-to-one and onto since it is invertible. Just as $T : V \to V$ is the linear map given by left-multiplication by A, $T^{-1} : V \to V$ is the linear map given by left-multiplication by A^{-1} as demonstrated in (b).

4. $A = \begin{bmatrix}-1&1&3\\2&1&-1\\1&2&2\end{bmatrix} \sim \begin{bmatrix}-1&1&3\\0&3&5\\1&2&2\end{bmatrix} \sim \begin{bmatrix}-1&1&3\\0&3&5\\0&3&5\end{bmatrix} \sim \begin{bmatrix}-1&1&3\\0&3&5\\0&0&0\end{bmatrix}$, so $\left\{\left[\begin{smallmatrix}4\\-5\\3\end{smallmatrix}\right]\right\}$ is a basis for $\mathrm{Nul}\,A$. In other words, the row space of A is the plane $4x - 5y + 3z = 0$.

5.

(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)	(i)	(j)
T	F	T	T	F	F	T	T	T	F

Although you are not expected to explain your answers to True/False questions, the following remarks may help to understand the solution key:

(a) As shown in class, any vector of the form $A\mathbf{x} \in \mathbb{R}^m$ is a linear combination of the n columns of A with weights given by the entries of $\mathbf{x} \in \mathbb{R}^n$.

(b) An example of an inconsistent linear system is $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

(c) If 0 is an eigenvalue for A, then a corresponding eigenvector $\mathbf{v} \neq \mathbf{0}$ lies in $\operatorname{Nul} A$.

(d) If $A\mathbf{v} = \lambda\mathbf{v}$ where $\mathbf{v} \neq \mathbf{0}$, then $A^2\mathbf{v} = A(A\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}$.

(e) One counterexample is provided by the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6 \in \mathbb{R}^3$ given by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ respectively.

(f) One counterexample is provided by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix}$ in $\mathbb{R}^3$.

(g) Since $\operatorname{Col} A$ has dimension 3, $\operatorname{Row} A$ must also have dimension 3.

(h) If $A\mathbf{v} = \lambda\mathbf{v}$, then $AB\mathbf{v} = BA\mathbf{v} = B(\lambda\mathbf{v}) = \lambda B\mathbf{v}$.

(i) If $\det\begin{bmatrix} a_{ij} & a_{i\ell} \\ a_{kj} & a_{k\ell} \end{bmatrix} \neq 0$, then rows i and k are linearly independent; also columns j and ℓ are linearly independent. Either way, this forces A to have rank at least two.

(j) One counterexample is $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. If you pick an example 'at random', the chance of A and B commuting is essentially 0%.