University of Wyoming — Math 2250—Fall 2023 — Department of Mathematics

Elementary Linear Algebra

Final Examination

December, 2023

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Total value of questions: 100 points (plus 10 bonus points). Time permitted: 120 minutes.

1. *(15 points)* Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Find a 2×2 matrix A having eigenvalues 1 and 2, with eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

2. *(30 points)* Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} x+2y-w \\ y+z+w \\ x-z+2w \\ 2x+3y+2w \end{bmatrix}.$$

(a) Write down the matrix A defining T (so that $T(\mathbf{v}) = A\mathbf{v}$).

(b) Find a basis for the null space of A.

(c) Is *every* vector in $\mathbb{R}^4$ of the form $T(\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^4$? Justify your answer.

(d) What is the rank of A?

(e) Is T invertible? Explain.

3. *(20 points)* Let V be the vector space of all 2×3 matrices with real entries, and let $T : V \to V$ be the linear transformation defined by $T(M) = AM$ where $A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

(a) What is the dimension of V? State a basis for V.

(b) Find M such that $T(M) = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \end{bmatrix}$.

(c) Is T one-to-one? Explain.

(d) Is T onto? Explain.

4. *(15 points)* The *row space* of the matrix

$$A = \begin{bmatrix} -1 & 1 & 3 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$$

is a two-dimensional subspace of $\mathbb{R}^3$, i.e. a plane through the origin. Find a linear equation of the form $ax + by + cz = 0$ defining this plane.

5. *(30 points)* Answer TRUE or FALSE to each of the following statements.

(a) If A is an $m \times n$ matrix, then the column space of A is $\{A\mathbf{x} : \mathbf{x} \in \mathbb{R}^n\}$.
_____*(True/False)*

(b) Every linear system $A\mathbf{x} = \mathbf{b}$ has at least one solution. _____*(True/False)*

(c) If 0 is an eigenvalue of A, then A is not invertible. _____*(True/False)*

(d) If λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2. _____*(True/False)*

(e) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_6\}$ be a set of vectors in $\mathbb{R}^n$. If the subsets $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and $\{\mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6\}$ are both linearly independent, then S must be linearly independent.
_____*(True/False)*

(f) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ be a set of vectors in $\mathbb{R}^n$. If $m > n$, then $\operatorname{Span} S$ must equal $\mathbb{R}^n$. _____*(True/False)*

(g) If three of the columns of a 3×6 matrix A are linearly independent, then the rows of A must be linearly independent. _____*(True/False)*

(h) If A and B are two $n \times n$ matrices satisfying $AB = BA$, and $\mathbf{v}$ is an eigenvector of A, then $B\mathbf{v}$ is an eigenvector of A. _____*(True/False)*

(i) Let A be a 9×9 matrix with (i, j)-entry equal to a_{ij}. If A has rank one, then $\det\begin{bmatrix} a_{ij} & a_{i\ell} \\ a_{kj} & a_{k\ell} \end{bmatrix} = 0$ for all i, j, k, ℓ between 1 and 9. _____*(True/False)*

(j) If A and B are $n \times n$ matrices, and B is diagonal, then $AB = BA$. (A *diagonal* matrix has all its nonzero entries on the main diagonal; that is, a matrix is diagonal if its (i, j)-entry is zero whenever $i \neq j$.) _____*(True/False)*