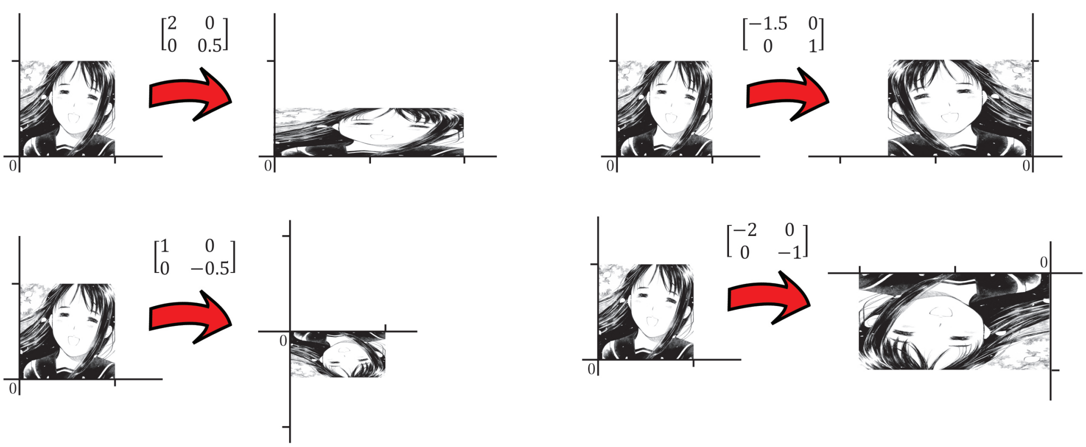
Eigenvalues and Eigenvectors

November, 2023

We have gained some geometric understanding of the action of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ in certain cases (including rotations, reflections, shears and dilations). Among those transformations most readily understood are those represented by diagonal matrices of the form $A = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$. Such a transformation $T(\mathbf{x}) = A\mathbf{x}$ maps the standard basis vectors $\mathbf{e}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\mathbf{e}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ to $a\mathbf{e}_1$ and $d\mathbf{e}_2$ respectively. Thus T stretches by a factor a in the horizontal direction, while also stretching by a factor d in the vertical direction. (To say T 'stretches' in the horizontal direction is perhaps only accurate for $a > 1$; if $0 < a < 1$ then T actually shrinks in the horizontal direction; if $a < 0$ then T reverses the horizontal direction; and if $a = 0$ then T flattens everything into the y-axis and so is not invertible. Similar observations apply for the vertical direction.) Examples:

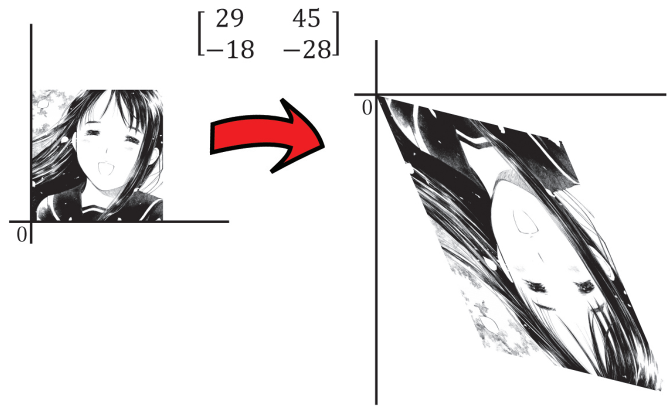


In each case, vectors in any direction other than horizontal or vertical do not retain their direction when T is applied. That is, for diagonal matrices of the form $A = \left[\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right]$ with $a \neq d$, the *only* vectors $\mathbf{x} \in \mathbb{R}^2$ for which $T(\mathbf{x})$ is a scalar multiple of $\mathbf{x}$, are horizontal and vertical vectors.

For a general linear transformation $T : V \to V$, if $T(\mathbf{x}) = \lambda\mathbf{x}$, then $\mathbf{x}$ is called an **eigenvector**, and $\lambda \in \mathbb{R}$ is the corresponding **eigenvalue**. (But we need a *nonzero* eigenvector $\mathbf{x} \neq \mathbf{0}$ in order to call λ an eigenvalue; for otherwise *every* scalar would qualify as an eigenvalue for $\mathbf{0}$). For diagonal matrices, the standard basis vectors (or scalar multiples thereof) are eigenvectors; and the diagonal entries are the eigenvalues. For more general linear transformations, some more work is required to determine the eigenvalues and eigenvectors (if any); and this information provides geometric insight into how the linear transformation acts.

Example

Consider the linear transformation $T:\mathbb{R}^2\to\mathbb{R}^2$ represented by the matrix $A=\left[\begin{smallmatrix}29&45\\-18&-28\end{smallmatrix}\right]$. Here is a suggestive illustration of how T acts:



This illustration is not to scale! (The vectors $T(\mathbf{e}_1)=\left[\begin{smallmatrix}29\\-18\end{smallmatrix}\right]$ and $T(\mathbf{e}_2)=\left[\begin{smallmatrix}45\\-28\end{smallmatrix}\right]$ are much too long to fit with the same scale; and the second parallelogram has such extreme angles that the figure is not easily recognized.) Since $\det A=-29\cdot 28+45\cdot 18=-2$, T doubles areas and reverses orientation (this much at least is roughly depicted by the illustration).

In order to find eigenvalues and eigenvectors for T, we must solve the equation $A\mathbf{x}=\lambda\mathbf{x}$, i.e. $(A-\lambda I)\mathbf{x}=\mathbf{0}$ for some nonzero $\mathbf{x}\in\mathbb{R}^2$; here we have the 2×2 identity matrix $I=I_2=\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]$. Thus eigenvectors for λ are vectors in the null space of $A-\lambda I$; and in order for λ to be an eigenvector of A (or of T), the null space of $A-\lambda I$ must have dimension at least 1. This means that $A-\lambda I$ is not invertible, i.e. $\det(A-\lambda I)=0$. This gives a polynomial equation (of degree n, if A is $n\times n$) called the **characteristic equation** for A; and $\det(A-\lambda I)$ is the **characteristic polynomial** of A. In our case, the characteristic polynomial is

$$\det(A-\lambda I)=\begin{vmatrix}29-\lambda & 45\\ -18 & -28-\lambda\end{vmatrix}=(29-\lambda)(-28-\lambda)-45(-18)=\lambda^2-\lambda-2=(\lambda+1)(\lambda-2).$$

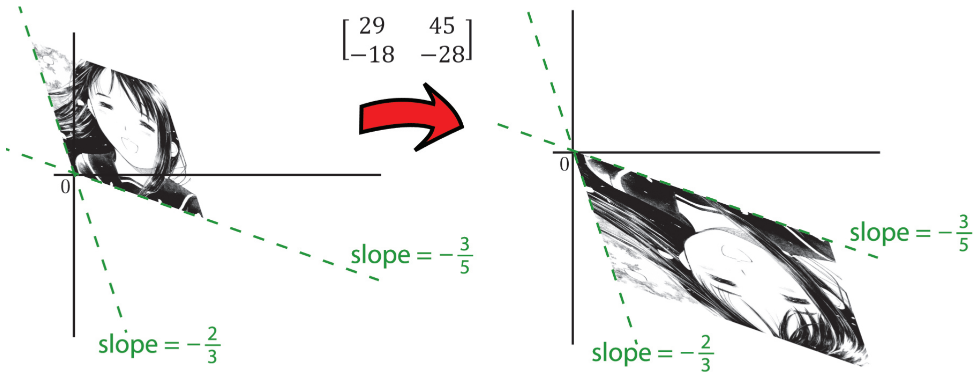
The eigenvalues are the roots of this polynomial, namely -1 and 2. An eigenvector for $\lambda_1=-1$ is any vector $\mathbf{v}_1$ spanning the null space of

$$A-\lambda_1 I \;=\; A+I \;=\; \begin{bmatrix}30&45\\-18&-27\end{bmatrix};$$

we may take $\mathbf{v}_1=\left[\begin{smallmatrix}3\\-2\end{smallmatrix}\right]$. An eigenvector for $\lambda_2=2$ is any vector $\mathbf{v}_2$ spanning the null space of

$$A-\lambda_2 I \;=\; A-2I \;=\; \begin{bmatrix}27&45\\-18&-30\end{bmatrix};$$

we may take $\mathbf{v}_2=\left[\begin{smallmatrix}5\\-3\end{smallmatrix}\right]$. Note that $A\mathbf{v}_1=-\mathbf{v}_1$: every vector in the line spanned by $\mathbf{v}_1$ (the line of slope $-\frac{2}{3}$ through the origin in $\mathbb{R}^2$) is reversed by T. Also $A\mathbf{v}_2=2\mathbf{v}_2$: every vector in the line spanned by $\mathbf{v}_2$ (the line of slope $-\frac{3}{5}$ through the origin in $\mathbb{R}^2$) is doubled by T. Moreover the *only* vectors on which T acts by simply scaling by a factor, are the vectors in these two lines.



Note that T scales areas by the factor $\det A = -2 = \lambda_1 \lambda_2$ in agreement with our previous observation.

Every vector $\mathbf{v} \in \mathbb{R}^2$ can be expressed as a linear combination of the new basis vectors $\mathbf{v}_1, \mathbf{v}_2$ as $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. With respect to the new basis, computing

$$A\mathbf{v} = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = -c_1\mathbf{v}_1 + 2c_2\mathbf{v}_2$$

is quite straightforward. Compare: using the standard basis one has $\mathbf{v} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2$ for some $a_1, a_2 \in \mathbb{R}$; and then

$$\begin{aligned} A\mathbf{v} &= A(a_1\mathbf{e}_1 + a_2\mathbf{e}_2) \\ &= a_1(29\mathbf{e}_1 - 18\mathbf{e}_2) + a_2(45\mathbf{e}_1 - 28\mathbf{e}_2) \\ &= (29a_1 + 45a_2)\mathbf{e}_1 - (18a_1 + 28a_2)\mathbf{e}_2 \,. \end{aligned}$$

When considering linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ for larger values of n, the advantage of a basis consisting of eigenvectors becomes increasingly apparent, with n terms in the expansion of $T(\mathbf{v})$ with respect to a basis of eigenvectors, as compared with n^2 terms (in the worst case) when another basis is used instead.