University of Wyoming

Math 2250—Fall 2023

Department of Mathematics

Elementary Linear Algebra

Solutions to HW1

(Due Monday, September 25, 2023 by 5:00pm on WyoCourses)

Instructions: Work by hand. Show your work. Always check your answers wherever feasible. Write clearly, using complete sentences where appropriate, and always using correct notation. For further instructions, see the syllabus and the FAQ's linked there. Total value of questions: 50 points.

1. (*10 points*) A linear system of three equations in six unknowns $x_1, x_2, \ldots, x_6$ is represented by the matrix

$$\begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & \end{matrix} \\ \left[\begin{array}{cccccc|c} 0 & 1 & 0 & 2 & -1 & 0 & -4 \\ 0 & 0 & 1 & 1 & 5 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 \end{array}\right]. \end{array}$$

(a) Which of the unknowns are *basic* variables? and which are *free* variables? (see p.18).

The basic variables are x_2, x_3, x_6, corresponding to the pivot columns. The other variables x_1, x_4, x_5 are the free variables.

(b) Write down the general solution of the linear system in terms of parameters r, s, t.

Assign the free variables to the parameters, viz. $x_1 = r$, $x_3 = s$, $x_6 = t$. The system takes the form

$$\begin{aligned} x_2 \quad + 2s - t &= -4 \\ x_3 + s + 5t &= 3 \\ x_6 &= 6 \end{aligned}$$

so that $(x_1, x_2, x_3, x_4, x_5, x_6) = (r, -4-2s+t, 3-s-5t, s, t, 6)$.

2. (*10 points*) You are given the matrix

$$A = \begin{bmatrix} 2 & 4 & -1 & -6 & 1 & 14 \\ 1 & 2 & 3 & -3 & 4 & 7 \\ 1 & 2 & -1 & -3 & 0 & 7 \end{bmatrix}.$$

Using a sequence of elementary row operations, reduce your answer to reduced row echelon form.

$$\begin{bmatrix} 2 & 4 & -1 & -6 & 1 & 14 \\ 1 & 2 & 3 & -3 & 4 & 7 \\ 1 & 2 & -1 & -3 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -3 & 4 & 7 \\ 2 & 4 & -1 & -6 & 1 & 14 \\ 1 & 2 & -1 & -3 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -3 & 4 & 7 \\ 0 & 0 & -7 & 0 & -7 & 0 \\ 1 & 2 & -1 & -3 & 0 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & -3 & 4 & 7 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -3 & 0 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -3 & 4 & 7 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & -3 & 4 & 7 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -3 & 1 & 7 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Check: By inspection, the vectors $v_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $v_4 = \begin{bmatrix} -7 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

are null vectors of the reduced row-echelon form. We check that $Av_1 = Av_2 = Av_3 = Av_4 = 0$.

3. (*10 points*) Consider the matrix $A = \left[\begin{smallmatrix} 1 & 0 \\ 3 & 1 \end{smallmatrix}\right]$. Evaluate A^2, A^3 and A^{100}.

$A^2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}$, $A^{100} = \begin{bmatrix} 1 & 0 \\ 300 & 1 \end{bmatrix}$.

A formal proof that $A^n = \begin{bmatrix} 1 & 0 \\ 3n & 1 \end{bmatrix}$ can be given by mathematical induction but we omit this in favor of simple pattern recognition.

4. (*10 points*) Consider the matrix $B = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Evaluate B^2, B^3, B^4, B^5 and B^{99}.

$B^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $B^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B^5 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $B^{99} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The sequence of powers of B repeats every fourth term.

5. (*10 points*) Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Find the set of all matrices that commute with A. (To do this, you must find all matrices X satisfying $AX = XA$ by solving the appropriate linear system.)

A matrix $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ commutes with A iff $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, iff $\begin{bmatrix} x+2z & y+2w \\ z & w \end{bmatrix} = \begin{bmatrix} x & 2x+y \\ z & 2z+w \end{bmatrix}$. This is a system of four linear equations in four unknowns, whose solutions are $\boxed{X = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = aI + bA,\ a,b \in \mathbb{R}.}$