Math 2200-01 (Calculus I) Spring 2020

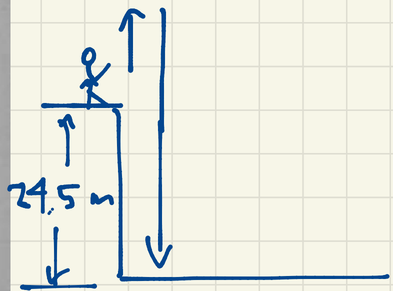
Book 2

Suppose a stone is thrown vertically upward from the edge of a cliff on Earth with an initial velocity of 19.6 m/s from a height of 24.5 m above the ground. The height (in meters) of the stone above the ground t seconds after it is thrown is $s(t) = -4.9t^2 + 19.6t + 24.5$.

a. Determine the velocity v of the stone after t seconds.
b. When does the stone reach its highest point?
c. What is the height of the stone at the highest point?
d. When does the stone strike the ground?
e. With what velocity does the stone strike the ground?
f. On what intervals is the speed increasing?

Sec 3.6 #24.

Mar 2



$s(t) = -4.9t^2 + 19.6t + 24.5$, $0 \le t \le 5$.

height of the stone above the ground in meters, at time t (in seconds).

(a) $v(t) = s'(t) = -9.8t + 19.6$, $0 \le t < 5$.

velocity at time t (in m/sec).

Note: $s(0) = 24.5$ m is the initial height;

$v(0) = 19.6$ m/sec is the initial velocity. In this problem, the motion is vertical with the positive direction being upwards.

(b) The stone reaches its highest point at the moment when the velocity changes sign from positive (upwards) to negative (downwards). At this moment the instantaneous velocity is zero. Solve $v(t) = -9.8t + 19.6 = 0$ to find $t = 2$ sec.

(c) The maximum height is $s(2) = 44.1$ m.

(d) The stone strikes the ground when $s(t) = -4.9t^2 + 19.6t + 24.5 = 0 = -4.9(t^2 - 4t - 5)$

$= -4.9(t-5)(t+1)$

This has two roots $t=-1, 5$ sec. But since $t \geq 0$, we must have $t = 5$ sec as the time when the stone hits the ground.

(e) The stone hits the ground with velocity $v(5) = -29.4$ m/sec (i.e. downwards at a speed of 29.4 m/sec).

(f) Speed is increasing during the time interval $2 < t < 5$ seconds.

<u>Remark</u> $a(t) = v'(t) = s''(t) = -9.8 \text{ m/sec}^2$ is constant.

<u>Sec 3.7 Chain Rule</u> Eg. find $\frac{d}{dx} \sin(e^x)$.

In general if $f(x) = g(h(x))$ and we know g', h', how do we find f' ?

In other words, if

$$x \xmapsto{h} u \xmapsto{g} y$$

x: independent variable

u: intermediate variable

y: dependent variable

As an example, think of $u = e^x$, $y = \sin u$

Small changes Δx in x give rise to small changes Δu in u, giving small changes Δy in y.

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

This refers to average rates of change. To get instantaneous rates of change, let $\Delta x \to 0$ so $\Delta u \to 0$ and $\Delta y \to 0$ giving

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

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Eg. $\frac{d}{dx}\sin(e^x) = e^x\cos(e^x)$

$$x \longmapsto u = e^x \longmapsto y = \sin u = \sin(e^x)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \underbrace{\cos u}_{\frac{dy}{du}} \cdot \underbrace{e^x}_{\frac{du}{dx}} = e^x\cos(e^x)$$

Eg. $\frac{d}{dx}(x^2+1)^3 = \frac{d}{dx}(x^6+3x^4+3x^2+1) = 6x^5+12x^3+6x$ (OLD WAY)

$\frac{d}{dx}(x^2+1)^3 = 3(x^2+1)^2\cdot 2x = 6x(x^2+1)^2$ (CHAIN RULE - NEW WAY)

CHECK: $6x(x^2+1)^2 = 6x(x^4+2x^2+1) = 6x^5+12x^3+6x$

agrees!

Rewriting this in function notation:

$$f(x) = g(h(x))$$

$$f'(x) = \underbrace{3(x^2+1)^2}_{g'(u)}\cdot\underbrace{2x}_{h'(x)} = g'(h(x))\,h'(x)$$

$$\begin{tikzcd} x \arrow[r, mapsto, "h"] \arrow[rr, bend right, "f"'] & u = x^2+1 \arrow[r, mapsto, "g"] & y = u^3 = (x^2+1)^3 \end{tikzcd}$$

$h(x) = x^2+1$

$h'(x) = 2x$

$g(u) = u^3$

$g'(u) = 3u^2$

(a) $h(x) = f(g(x))$

$h'(x) = f'(g(x))\,g'(x)$

$h'(1) = f'(g(1))\,g'(1)$

$= f'(4)\cdot 9 = 7\cdot 9 = 63.$

(b) $h'(2) = f'(g(2))\,g'(2)$

$= f'(1)\cdot 7 = (-6)\cdot 7 = -42$

26. Derivatives using tables Let $h(x) = f(g(x))$ and $k(x) = g(g(x))$. Use the table to compute the following derivatives.

a. $h'(1)$ **b.** $h'(2)$ **c.** $h'(3)$ **d.** $k'(3)$ **e.** $k'(1)$ **f.** $k'(5)$

x	1	2	3	4	5
$f'(x)$	−6	−3	8	7	2
$g(x)$	4	1	5	2	3
$g'(x)$	9	7	3	−1	−5

(c) $h'(3) = f'(g(3))\,g'(3)$

$= f'(5)\cdot 3 = 2\cdot 3 = 6$

(e) $k(x) = g(g(x))$

$k'(x) = g'(g(x))\,g'(x)$

$k'(1) = g'(g(1))\,g'(1)$

$= g'(4)\cdot 9 = -1\cdot 9 = -9$

(d) $k'(3) = g'(g(3))\,g'(3)$

$= g'(5)\cdot 3$

$= -5\cdot 3 = -15$

(f) $k'(5) = g'(g(5))\,g'(5)$

$= g'(3)(-5) = 3\cdot(-5) = -15.$

Eg. $\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x)(\sin x) = \sin x\cos x + \cos x\sin x = 2\sin x\cos x$ (OLD WAY)

$\frac{d}{dx}\sin^2 x = \frac{d}{dx}(\sin x)^2 = 2\sin x\cos x$ (NEW WAY - CHAIN RULE)

$$\frac{d}{dx}\sin\left(\frac{3x}{x^2+1}\right) = \cos\left(\frac{3x}{x^2+1}\right)\cdot\frac{(x^2+1)\cdot 3 - 3x(2x)}{(x^2+1)^2} = \frac{-3x^2+3}{(x^2+1)^2}\cos\left(\frac{3x}{x^2+1}\right)$$

$$= 3\,\frac{1-x^2}{(x^2+1)^2}\cos\left(\frac{3x}{x^2+1}\right)$$

Mar 4

$x \longmapsto u \longmapsto y \qquad \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$

$x \longmapsto u \longmapsto v \longmapsto y \qquad \frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx}$

Eg. $\frac{d}{dx}\sqrt{\tan(x^2)}$

$x \longmapsto u = x^2 \longmapsto v = \tan u \longmapsto y = \sqrt{v}$

Note: $\frac{d}{dv}\sqrt{v} = \frac{d}{dv} v^{1/2} = \frac{1}{2}v^{-\frac{1}{2}} = \frac{1}{2\sqrt{v}}$

$\frac{du}{dx} = 2x \qquad \frac{dv}{du} = \sec^2 u \qquad \frac{dy}{dv} = \frac{1}{2\sqrt{v}}$

$$\frac{dy}{dx} = \frac{dy}{dv}\cdot\frac{dv}{du}\cdot\frac{du}{dx} = \frac{1}{2\sqrt{v}}\cdot\sec^2 u\cdot 2x = \frac{2x\sec^2(x^2)}{2\sqrt{\tan(x^2)}} = \frac{x\sec^2(x^2)}{\sqrt{\tan(x^2)}}$$

OR: $\frac{d}{dx}\sqrt{\tan(x^2)} = \frac{1}{2\sqrt{\tan(x^2)}}\cdot\sec^2(x^2)\cdot 2x = \frac{x\sec^2(x^2)}{\sqrt{\tan(x^2)}}$

If $f(x) = \sec(3x+1)$, find $f'(x)$ and $f''(x)$.

Recall: $\frac{d}{dt}\sec t = \sec t \tan t$

$\frac{d}{dt}\tan t = \sec^2 t$

$$f'(x) = \sec(3x+1)\tan(3x+1)\cdot 3 = 3\sec(3x+1)\tan(3x+1)$$

$$f''(x) = 3\underbrace{\left(3\sec(3x+1)\tan(3x+1)\right)}_{\frac{d}{dx}\sec(3x+1)}\tan(3x+1) + 3\sec(3x+1)\underbrace{\left(\sec^2(3x+1)\cdot 3\right)}_{\frac{d}{dx}\tan(3x+1)}$$

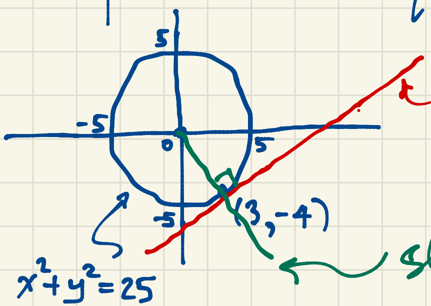
$$= 9\sec(3x+1)\tan^2(3x+1) + 9\sec^3(3x+1).$$

$\sec^2 t = 1 + \tan^2 t$

$\frac{d}{dx}\tan(3x+1) = \sec^2(3x+1)\cdot 3$

Sec 3.8: Implicit Differentiation

Example: Find the equation of the tangent line to the circle $x^2+y^2=25$ at the point $(3,-4)$.



the normal line to the circle at $(3,-4)$ (ie. perpendicular) is $y = -\frac{4}{3}x$.

Take $\frac{d}{dx}$ of both sides:

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}25$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\left.\frac{dy}{dx}\right|_{(3,-4)} = \frac{3}{4}$$

$$\frac{dy^2}{dx} = \frac{dy^2}{dy}\cdot\frac{dy}{dx} = 2y\frac{dy}{dx}$$

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The tangent line to the circle at $(3,-4)$ is

$$y+4 = \frac{3}{4}(x-3) \quad \text{i.e.} \quad y = \frac{3}{4}x - \frac{25}{4}$$

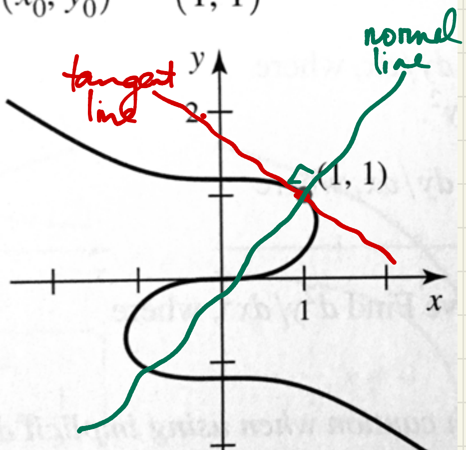
Alternatively, the circle $x^2+y^2=25$ has $y=\pm\sqrt{25-x^2}$ so it consists of an upper semicircle $y=\sqrt{25-x^2}$ and a lower semicircle $y=-\sqrt{25-x^2}$. Our point $(3,-4)$ is on the lower semicircle. If $f(x)=-\sqrt{25-x^2}=-(25-x^2)^{1/2}$ then $f'(x)=-\frac{1}{2}(25-x^2)^{-1/2}(-2x)$ $=\frac{x}{\sqrt{25-x^2}}$ So $f'(3)=\frac{3}{\sqrt{25-9}}=\frac{3}{4}$. So the tangent line is $y=\frac{3}{4}x-\frac{25}{4}$ as before.

79–82. Visualizing tangent and normal lines

a. *Determine an equation of the tangent line and the normal line at the given point (x_0, y_0) on the following curves. (See instructions for Exercises 73–78.)*

b. *Graph the tangent and normal lines on the given graph.*

79. $3x^3 + 7y^3 = 10y;$ $(x_0, y_0) = (1, 1)$



Check first that $(1,1)$ lies on the curve!

$3\cdot 1^3 + 7\cdot 1^3 = 10\cdot 1.$

$$\frac{d}{dx}\left(3x^3+7y^3\right)=\frac{d}{dx}10y$$

$$9x^2+21y^2\frac{dy}{dx}=10\frac{dy}{dx}$$

Substitute $(1,1)$:

$m=\left.\frac{dy}{dx}\right|_{(1,1)}$ satisfies

$$9+21m=10m$$

$$9=-11m$$

$$m=-\frac{9}{11}$$

The tangent line to the curve at $(1,1)$ is

$$y-1=-\frac{9}{11}(x-1)$$

ie. $y=-\frac{9}{11}x+\frac{20}{11}$

The normal line at $(1,1)$ has slope $-\frac{1}{m} = \frac{11}{9}$ so the normal line at $(1,1)$ is

$$y-1 = \frac{11}{9}(x-1) \quad \text{i.e.} \quad y = \frac{11}{9}x - \frac{2}{9}.$$

The power rule $\frac{d}{dx}x^n = nx^{n-1}$ was first explained for $n = 0,1,2,3,\dots$ but it works for arbitrary exponent. Here's why:

$$\frac{d}{dx}\frac{1}{x} = -\frac{1}{x^2}$$

If $n = -1,-2,-3,-4,\dots$ then $-n = 1,2,3,4,\dots$

$$\frac{d}{dx}x^n = \frac{d}{dx}\frac{1}{x^{-n}} = \frac{x^{-n}\cdot 0 - 1\cdot(-n)x^{-n-1}}{(x^{-n})^2} = n\frac{x^{-n-1}}{x^{-2n}} = nx^{n-1}$$

If $n = \frac{a}{b}$ where a,b are integers and $y = x^n = x^{a/b}$ so $y^b = x^a$ so

$$by^{b-1}\frac{dy}{dx} = ax^{a-1} \text{ and } \frac{dy}{dx} = \frac{a}{b}\frac{x^{a-1}}{y^{b-1}} = \frac{a}{b}\frac{x^{a-1}}{x^{\frac{a}{b}(b-1)}} = \frac{a}{b}x^{\frac{a}{b}-1} = nx^{n-1}$$