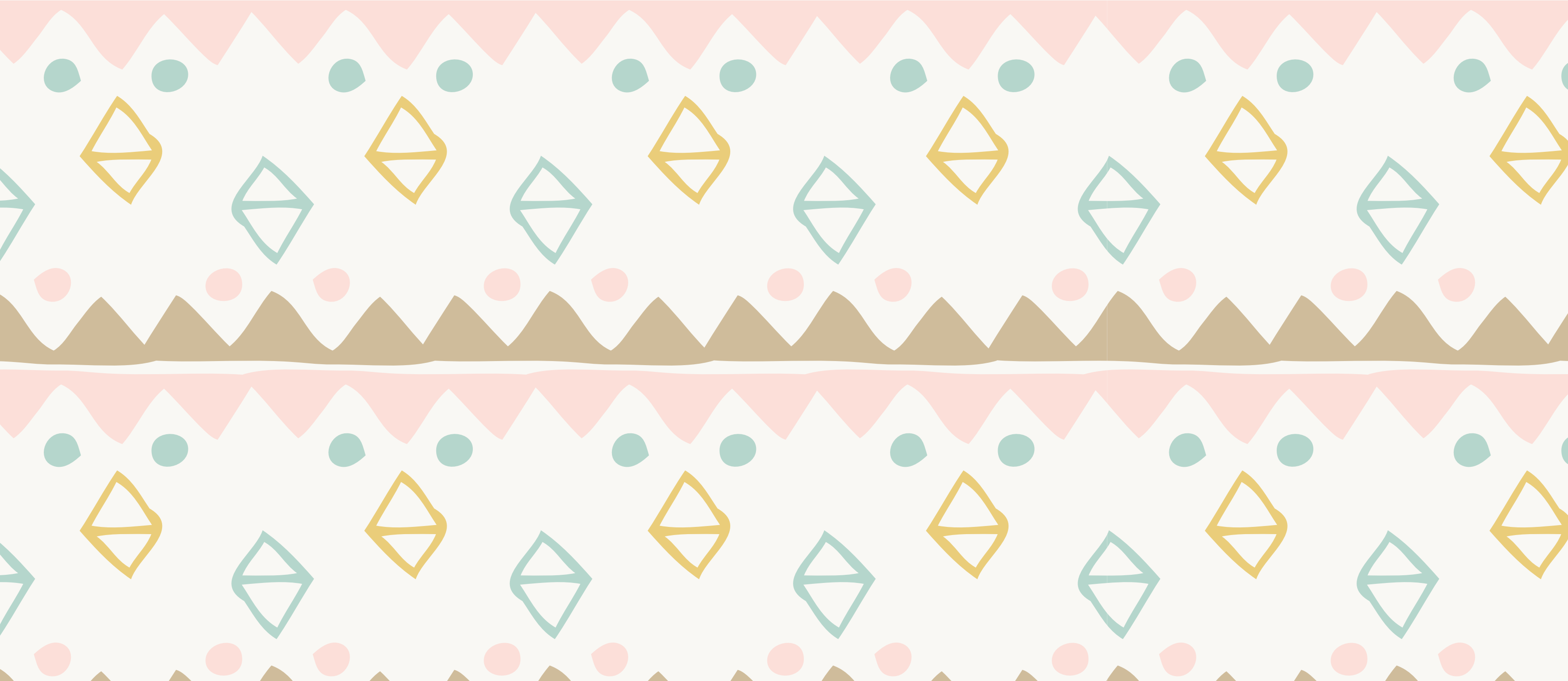
Math 2200-01 (Calculus I) Spring 2020

Book 1



Jan 27

Calculus I : Single-variable calculus $y = f(x)$ for example (one input variable x, one output variable). Derivatives (rates of change) : <u>differential</u> calculus.

Calculus II : also single-variable. <u>Integral</u> calculus.

Calculus III : multivariable i.e. several input variables and/or several output variables

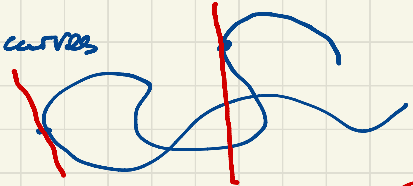
<u>eg</u>. position $(x(t), y(t), z(t))$ of an object at time t: one input t, three output variables $x(t), y(t), z(t)$.

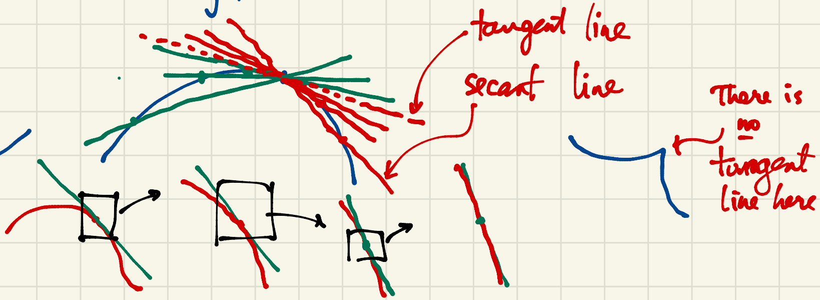
Eg. Temperature in this room as a function of position $T(x,y,z)$ (three inputs x,y,z; one output T)

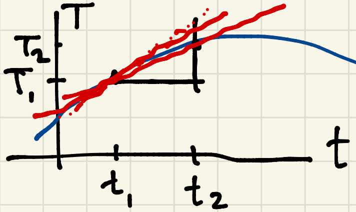
<u>Eg</u>. Wind velocity as a function of position : three inputs x,y,z; three outputs are the components of wind velocity.

Jan 28

Tangent lines to curves







Temperature T as a function of time t

During the time interval $[t_1, t_2]$ i.e. $t_1 \le t \le t_2$
the temperature rises from T_1 to T_2.

The average rate of change of temperature during this time interval is

$$\frac{\Delta T}{\Delta t} = \frac{T_2 - T_1}{t_2 - t_1}$$

← change in temperature

← time elapsed.

= slope of the secant line from (t_1, T_1) to (t_2, T_2) on the graph.

We want to understand the instantaneous rate of change of temperature at time t_1. To determine this, first consider the average rate of change over smaller and smaller time intervals $[t_1, t_2]$ where we take $t_2 \to t_1$ (t_2 gets closer and closer to t_1).

Jan 29

Eg.

t_2	$\frac{T_2 - T_1}{t_2 - t_1}$
4	2 degrees/hour
3.2	2.17
3.1	2.19
3.001	2.197
2.9	2.204
2.7	2.23
2	2.31

In my example, $t_1 = 3$.

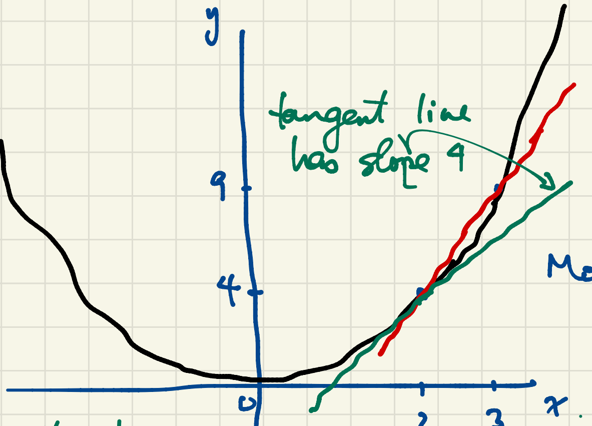
The limit is 2.2.

(The temperature at 3pm is changing at a rate of 2.2 degrees per hour.

We write $$\lim_{t_2 \to 3} \frac{T_2 - T_1}{t_2 - t_1} = 2.2$$

(the limit of $\frac{T_2 - T_1}{t_2 - t_1}$ is 2.2 as t_2 approaches 3).

A second example using a polynomial function $y = f(x) = x^2$. Find the rate of change of y with respect to x at $x = 2$.



The secant line joining the points $(2,4)$ and $(3,9)$ on the curve has slope

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5.$$

More generally, if we join the points $\overset{f(2)=4}{(2,4)}$ and $(x, f(x))$ on the curve, the secant line has slope

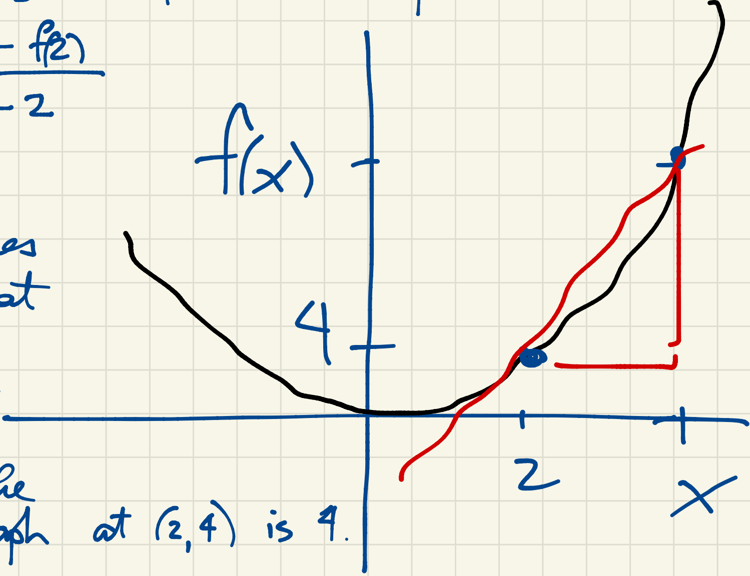
$$\frac{\Delta y}{\Delta x} = \frac{f(x) - f(2)}{x-2}$$

The tangent line at $(2,4)$ is $y - 4 = 4(x-2)$ i.e. $y = 4x - 4$.

x	$\dfrac{f(x)-f(2)}{x-2}$
3	5
2.5	4.5
2.1	4.1
2.01	4.01
1.99	3.99
1.9	3.9
1.5	3.5
1	3

Based on the table of values we guess that

$$\lim_{x \to 2} \frac{f(x) - f(2)}{x-2} = 4.$$



i.e. the slope of the tangent line to the graph at $(2,4)$ is 4.

If a function has a sufficiently nice formula eg. polynomial, then we have algebraic rules that provide definite ways to evaluate limits, eliminating guesswork based on the graph or table of values.

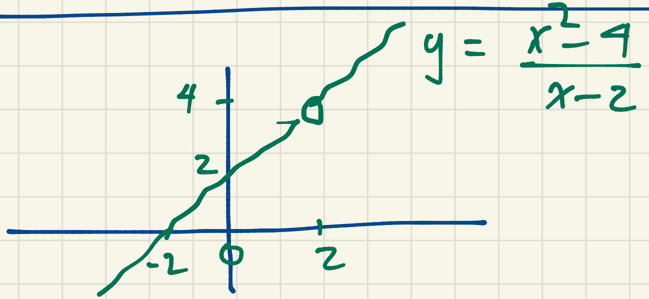
Eg. Find the slope of the tangent line to the graph of $y=x^2$ at $(2,4)$.

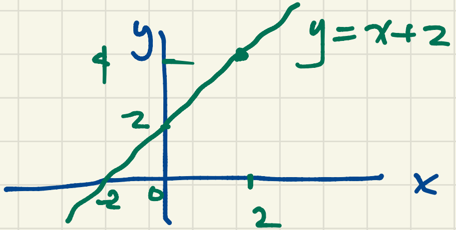
Solution The secant line from $(2,4)$ to $(x, f(x)) = (x, x^2)$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2 \quad \text{for } x \neq 2.$$

The slope of the tangent line is

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 2+2 = 4.$$





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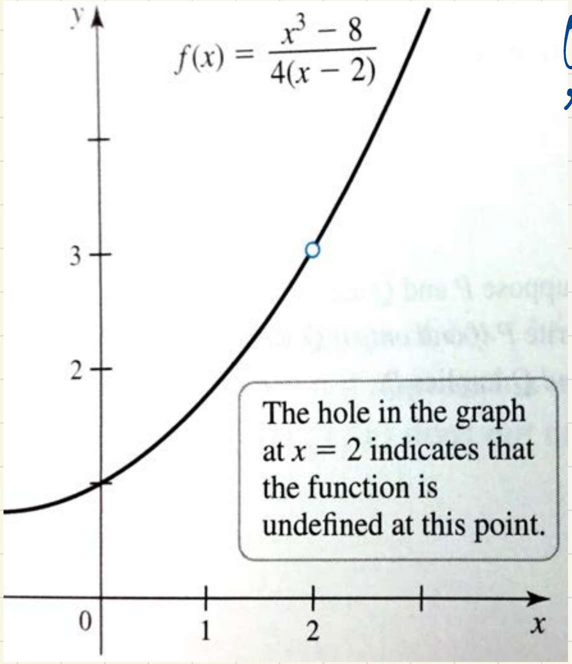
Both of these functions satisfy $\lim_{x\to 2} f(x) = 4$



The hole in the graph at $x = 2$ indicates that the function is undefined at this point.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3-8}{4(x-2)} = 3$$

Feb 10

Compare: Friday's quiz

$$\lim_{x\to 3} \frac{1}{(x-3)^2} = \infty$$

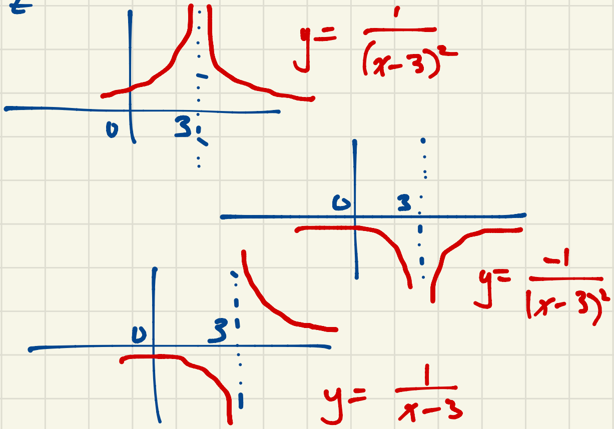
$$\lim_{x\to 3} \frac{-1}{(x-3)^2} = -\infty$$

$\lim_{x\to 3} \frac{1}{x-3}$ does not exist

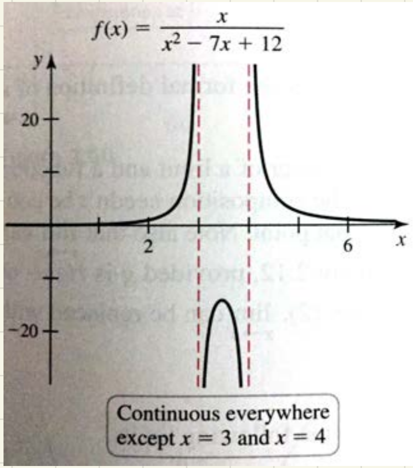
$$\lim_{x\to 3^+} \frac{1}{x-3} = \infty \qquad \lim_{x\to 3^-} \frac{1}{x-3} = -\infty$$

$$\lim_{x\to \infty} \frac{1}{x-3} = 0$$

$$\lim_{x\to -\infty} \frac{1}{x-3} = 0$$

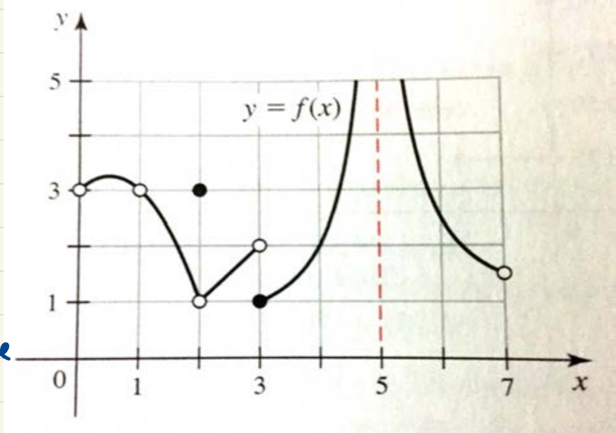


Sec 2.6



A function f is continuous at a if $\lim_{x\to a} f(x) = f(a)$.
Explicitly, this requires that
(i) f must be defined at a, ie. f(a) exists;
(ii) f must have a limit at a; and
(iii) the values in (i) and (ii) must agree.

Eg. for the function f on the right,



- f is discontinuous at 5;
- f " " " 3.

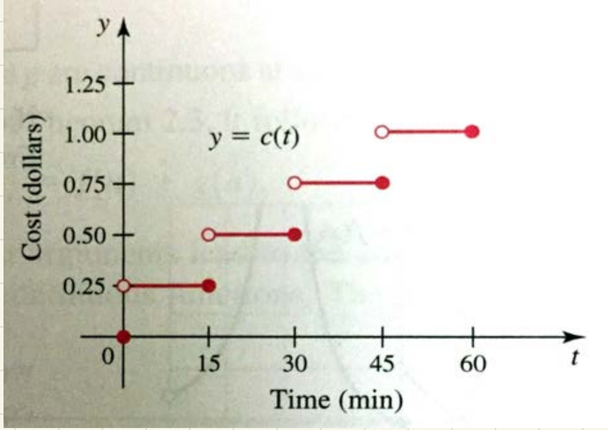
  $f(3) = 1$, $\lim_{x\to 3} f(x)$ does not exist.

- f is not continuous at 2.

  $f(2) = 3$, $\lim_{x\to 2} f(x) = 1$ but these two values do not agree!

- f is discontinuous at 1. Although $\lim_{x\to 1} f(x) = 3$, f is not defined at 1.

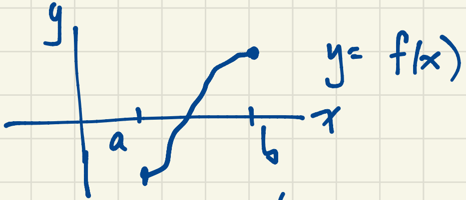
f is continuous on $(0, 7)$ ie. $0 < x < 7$ except at 1, 2, 3, 5.



Eg. the cost of parking at a meter is 25¢ for each 15 minutes. The cost c(t) as a function of time is discontinuous at $t = 0, 15, 30, 45, 60, \cdots$ At each of these points of discontinuity, c is left-continuous (i.e. $\lim_{t\to a^-} f(t) = f(a)$) but not right-continuous (i.e. $\lim_{t \to a^+} f(t) \neq f(a)$).

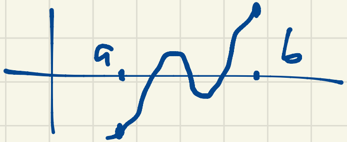
Feb 11

Why do we care about continuity?

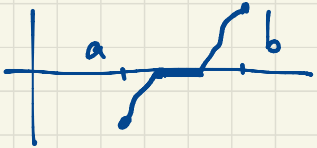


If f is continuous with $f(a) < 0$ and $f(b) > 0$ then there exists c, $a < c < b$, such that $f(c) = 0$. (Intermediate Value Theorem)

Remarks: The point c might not be unique i.e. there might be more than one c with this property.

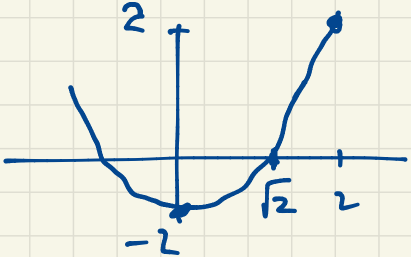


or



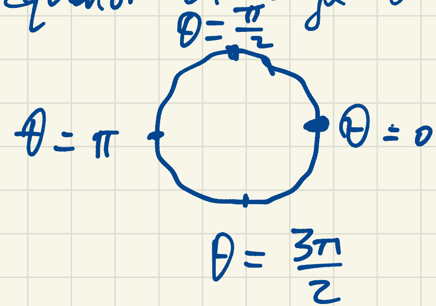
What is $\sqrt{2}$? Why does such a number exist? Consider $f(x) = x^2 - 2$.

f is continuous because it is a polynomial (See Sec 2.6). By the Intermediate Value theorem (since $f(0) < 0$, $f(2) > 0$) there exists c between 0 and 2 such that $f(c) = 0$. Later, as we'll see, there is only one such c. We call this value $\sqrt{2}$.



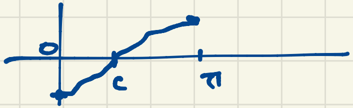
Another example: At this moment there are two points which are antipodes on the Earth's surface having exactly the same temperature.

Consider the equator and let $T(\theta)$, $0 \le \theta < 2\pi$, be the temperature on the equator at angle θ with respect to 0° longitude (i.e. θ is longitude).

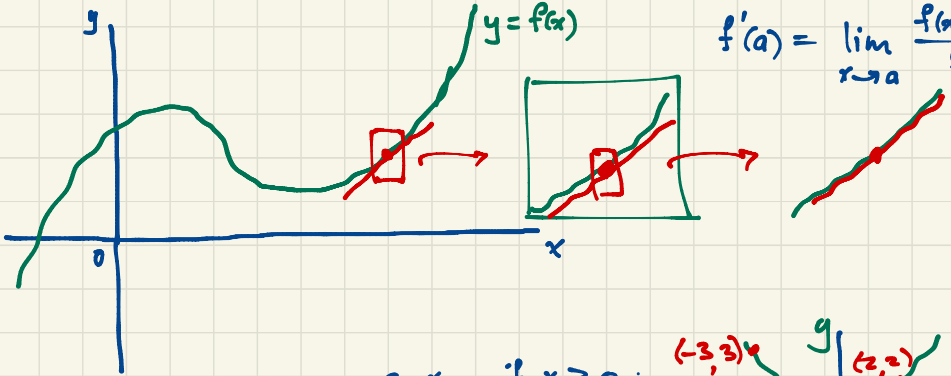


$f(\theta) = T(\theta+\pi) - T(\theta)$ = difference in temperature between longitude θ and its antipode (at $\theta+\pi$).

If $f(0) < 0$ i.e. $T(\pi) < T(0)$ then $f(\pi) > 0$.



There exists c, $0 < c < \pi$ such that $f(c) = 0$. i.e. $T(c) = T(c+\pi)$.



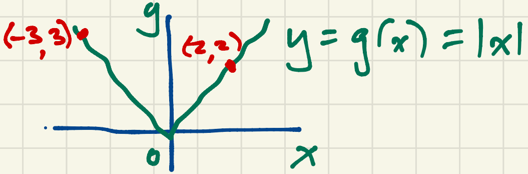
$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

$(h = \Delta x = x-a)$

$\Delta y = f(x) - f(a) = f(a+h) - f(a)$

Feb 14 ♡

Eg. $g(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$



$$g'(2) = \lim_{x\to 2}\frac{g(x)-g(2)}{x-2} = \lim_{x\to 2}\frac{|x|-2}{x-2} = \lim_{x\to 2}\frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1.$$

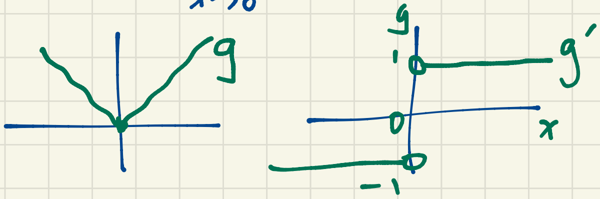
$$g'(-3) = \lim_{x\to -3}\frac{|x|-3}{x+3} = \lim_{x\to -3}\frac{-x-3}{x+3} = \lim_{x\to -3}(-1) = -1.$$

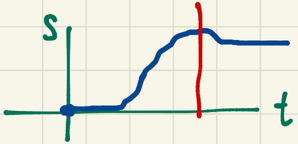
$g'(0) = \lim_{x\to 0}\frac{|x|-0}{x-0} = \lim_{x\to 0}\frac{|x|}{x}$ does not exist $\left(\lim_{x\to 0^+}\frac{|x|}{x} = 1\right.$ whereas $\left.\lim_{x\to 0^-}\frac{|x|}{x} = -1\right)$.

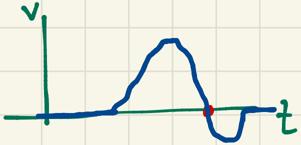
$g'(0)$ does not exist. $|x|$ is not differentiable at 0.

$g'(a) = \begin{cases} 1 & \text{if } a > 0; \\ -1 & \text{if } a < 0. \end{cases}$

(undefined if $a = 0$).







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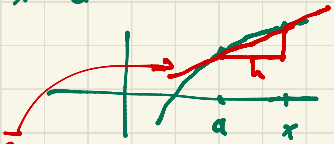
s = position (displacement)

t = time

v = velocity $\quad v(t) = s'(t)$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$$

$h = \Delta x = x - a$



secant line from $(a, f(a))$ to $(x, f(x))$ has slope $\frac{\Delta y}{\Delta x} = \frac{f(x) - a}{x-a}$

$= \frac{f(a+h) - f(a}{h}$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Eg. $f(x) = x^2$

$f(3) = 9$

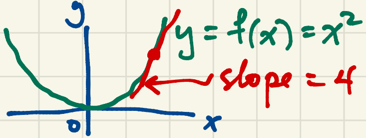
$f(-2) = 4$

$f(w) = w^2$

$f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$

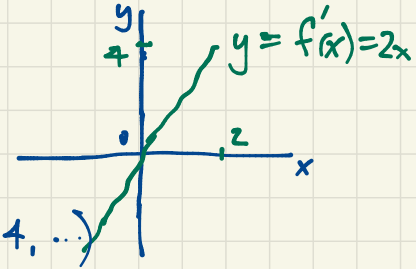
$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \lim_{h\to 0}\frac{x^2+2hx+h^2-x^2}{h} = \lim_{h\to 0}\frac{2hx+h^2}{h}$$

$$= \lim_{h\to 0}\frac{(2x+h)h}{h} = \lim_{h\to 0}(2x+h) = 2x.$$



The derivative of $f(x)=x^2$ with respect to x is $f'(x) = 2x$.

$$\boxed{\frac{dx^2}{dx} = 2x}$$



More generally, let n be a positive integer (i.e. n is one of $1, 2, 3, 4, \ldots$)

Consider the power function $f(x) = x^n$.

$$f'(a) = \lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1})}{x-a}$$

$$= \lim_{x\to a}\left(x^{n-1}+ax^{n-2}+a^2x^{n-3}+\cdots+a^{n-2}x+a^{n-1}\right)$$

$$= \underbrace{a^{n-1}+a^{n-1}+a^{n-1}+\cdots+a^{n-1}}_{n} = na^{n-1}$$

So $f'(x) = nx^{n-1}$

$$\frac{dx^n}{dx} = nx^{n-1}$$

Feb 18

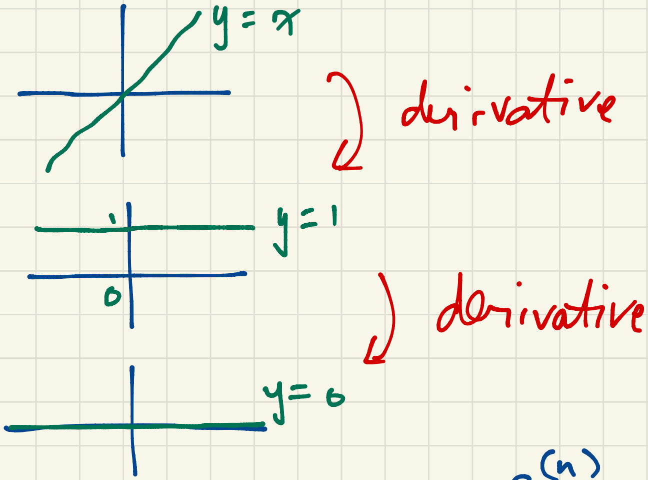
$$\frac{d}{dx}x^2 = 2x$$

$$\frac{dx^2}{dx}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{dx}{dx} = 1$$

$$\frac{d1}{dx} = 0$$



$$f \xrightarrow[\text{derivative}]{} f' \xrightarrow[\text{derivative}]{} f'' \xrightarrow[\text{derivative}]{} f''' \longrightarrow \cdots \longrightarrow f^{(n)}$$

nth derivative of f

$f^{(1)} = f' =$ (first) derivative of f

$f^{(2)} = f'' =$ second derivative of f etc.

Eg. if $f(x) = x^3$ then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

If c is constant then $\frac{d}{dx} cy = c \frac{dy}{dx}$ i.e. $(cf)' = cf'$

$(f+g)' = f' + g'$ i.e. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Now we can take the derivative of any polynomial eg.

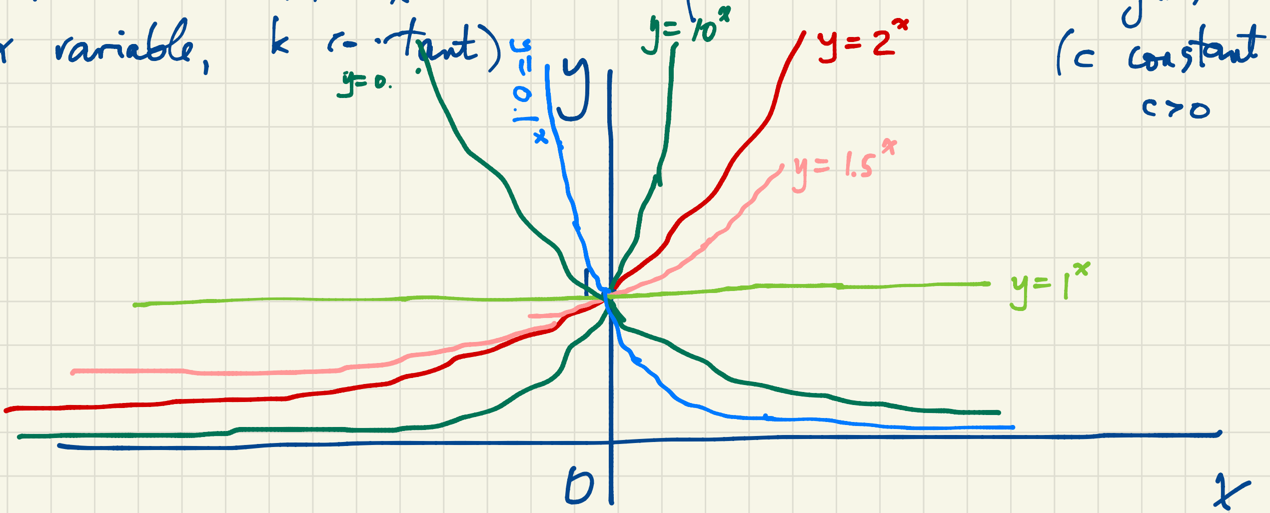
$$\frac{d}{dx}\left(7x^3 - 3x^2 - 5x + 11\right) = 21x^2 - 6x - 5$$

i.e. if $f(x) = 7x^3 - 3x^2 - 5x + 11$ then $f'(x) = 21x^2 - 6x - 5$.

Power function $f(x) = x^k$ vs. Exponential function $g(x) = c^x$

(x variable, k constant) (c constant, x variable)

$c > 0$



The function $f(x) = c^x$ exhibits exponential growth for $c > 1$ (faster growth than any power function) and exponential decay for $0 < c < 1$. As we vary the base c of the exponential, the curve $y = c^x$ passes through the intercept $(0,1)$ with varying slope, e.g. slope ≈ 0.693 when $c = 2$ and slope ≈ 1.099 when $c = 3$. We expect that for some c between 2 and 3, the curve $y = 2^x$ will pass through $(1,0)$ with slope exactly 1 (and this expectation can be justified using the Intermediate Value Theorem). Accordingly, we define e to be the unique constant for which the curve $y = e^x$ has tangent line of slope exactly 1 at the point $(0,1)$. Thus by definition, if $f(x) = e^x$, then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and e is the unique number with this property. (So we may take this as our definition of e.) It may be shown that $e \approx 2.71828\ldots$.

If $f(x)=e^x$ then $f'(0)=\lim\limits_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim\limits_{h\to 0}\frac{e^h-1}{h}=1$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{x+h}-e^x}{h}=\lim_{h\to 0}\frac{e^xe^h-e^x}{h}$$

$$=\lim_{h\to 0}e^x\cdot\frac{e^h-1}{h}=e^x\cdot 1=e^x$$

$$\boxed{\frac{d}{dx}e^x=e^x}$$

Feb 19

$$e^{a(x+h)}=e^{ax+ah}=e^{ax}\cdot e^{ah}$$

If $f(x)=e^{ax}$ (a constant) then

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{e^{a(x+h)}-e^{ax}}{h}=\lim_{h\to 0}\frac{e^{ax}e^{ah}-e^{ax}}{h}=\lim_{h\to 0}e^{ax}\cdot\frac{e^{ah}-1}{h}$$

$$=e^{ax}\lim_{h\to 0}\frac{e^{ah}-1}{h}=e^{ax}\lim_{t\to 0}\frac{e^t-1}{t/a}=ae^{ax}\lim_{t\to 0}\frac{e^t-1}{t}=ae^{ax}\cdot 1=ae^{ax}.$$

Substitute $t=ah$

i.e. $h=\frac{t}{a}$

Set $a=\ln 2 \iff e^a=2$.

$\iff (e^a)^x=2^x$

$\iff e^{ax}=2^x$

$$\boxed{\frac{d}{dx}e^{ax}=ae^{ax}}$$

Exponentials and logarithms are inverse functions

$$\begin{tikzcd} a \arrow[r, bend left, "\exp"] & 2 \arrow[l, bend left, "\ln"] \end{tikzcd}$$

$2^x=e^{(\ln 2)x}$

$\frac{d}{dx}2^x=(\ln 2)\,2^x$

$\frac{d}{dx}c^x=c^x\cdot\ln c$

Question: For which functions f does $f' = f$? ie. $\frac{dy}{dx} = y$?

Answer: Functions of the form $f(x) = ke^x$ where k is constant have $f'(x) = ke^x = f(x)$.
It turns out (later...) that these are the only solutions.

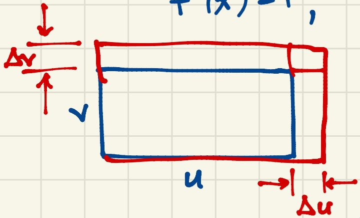
$$\frac{d}{dx} e^{4x+7} = \frac{d}{dx} e^7 \cdot e^{4x} = e^7 \cdot 4e^{4x} = 4e^{4x+7}.$$

$$a^{mn} = (a^m)^n$$
$$a^{m+n} = a^m \cdot a^n$$

Sec 3.4: Product Rule. $(f+g)' = f'+g'$ but $(fg)' \neq f'g'$.

Eg. $f(x) = x+3$, $g(x) = x^2$. $\quad f(x)g(x) = (x+3)x^2 = x^3+3x^2$

$f'(x) = 1$, $g'(x) = 2x$ $\quad (fg)'(x) = 3x^2+6x \neq f'(x)g'(x)$



Original area $A = uv$ at time t

New area $= (u+\Delta u)(v+\Delta v) = uv + u\Delta v + v\Delta u + (\Delta u)(\Delta v)$ at time $t+\Delta t$

Imagine a rectangle $u \times v$ which grows in time. If we increase length u by Δu and the width v by Δv then what is the change in area?

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$$\frac{\Delta A}{\Delta t} = \frac{u\Delta v + v\Delta u + \Delta u\Delta v}{\Delta t} = u\frac{\Delta v}{\Delta t} + \frac{\Delta u}{\Delta t}v + \underbrace{\Delta u}_{\downarrow \, 0} \cdot \underbrace{\frac{\Delta v}{\Delta t}}_{\searrow \frac{dv}{dt}}.$$

Let $\Delta t \to 0$. In the limit,

$$\frac{dA}{dt} = u\frac{dv}{dt} + \frac{du}{dt}v + 0$$

$$\frac{d}{dt}(uv) = u\frac{dv}{dt} + \frac{du}{dt}v$$ (the product rule)

$$(fg)' = f'g + fg'$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2}$$ (quotient rule)

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$$u = \frac{u}{v}\cdot v$$

$$\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{v}\cdot v\right) = \frac{d}{dt}\left(\frac{u}{v}\right)\cdot v + \frac{u}{v}\cdot\frac{dv}{dt}$$

$$\frac{1}{v}\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{v}\right) + \frac{u}{v^2}\frac{dv}{dt}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dt} - \frac{u}{v^2}\frac{dv}{dt}$$

$$\frac{d}{dx}(xe^x) = x\cdot e^x + e^x\cdot 1 = (x+1)e^x$$

slow motion:

$$\frac{d}{dx}(xe^x) = x\frac{de^x}{dx} + \frac{dx}{dx}e^x = xe^x + 1\cdot e^x = (x+1)e^x.$$

$$\frac{d}{dx}(x+3)x^2 = (x+3)\cdot 2x + 1\cdot x^2 = 3x^2 + 6x$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

$$\frac{d}{dx}(xe^{-x}) = 1\cdot e^{-x} + x(-e^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

Alternatively,

$$\frac{d}{dx}(xe^{-x}) = \frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x\cdot 1 - x\cdot e^x}{(e^x)^2} = \frac{1-x}{e^x} = (1-x)e^{-x}.$$

Eg. $$\frac{d}{dx}\frac{4x^2-3}{(x+1)^2} = \frac{(x+1)^2\cdot 8x - (4x^2-3)\cdot(2x+2)}{(x+1)^4}$$

$$= \frac{(x+1)\cdot 8x - (4x^2-3)\cdot 2}{(x+1)^3} = \frac{8x+6}{(x+1)^3}$$

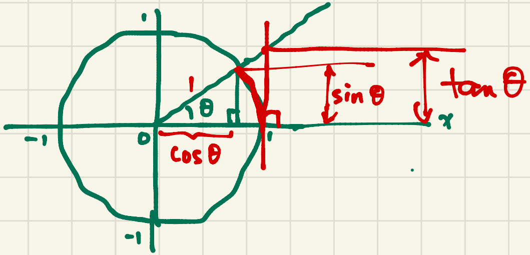
$$\frac{d}{dx}(x+1)^2 = \frac{d}{dx}(x^2+2x+1)$$

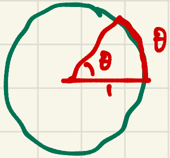
$$= 2x+2 = 2(x+1)$$

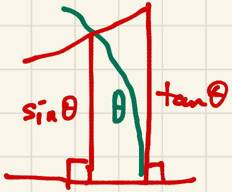
$50\% = 0.5$ says $50 \times \frac{1}{100} = 0.5$ $\left(\% = \frac{1}{100}\right)$

$180^\circ = 180 \times \frac{2\pi}{360} = \pi$ $\left({}^\circ = \frac{2\pi}{360} = \frac{\pi}{180} \right)$

Radian measure for angles is raw numbers with no units.







$$\sin\theta \le \theta \le \tan\theta = \frac{\sin\theta}{\cos\theta} \Rightarrow \cos\theta \le \frac{\sin\theta}{\theta} \le 1$$

$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ by the Squeeze Theorem since $\lim_{\theta\to 0} \cos\theta = \cos 0 = 1$ and $\lim_{\theta\to 1} 1 = 1$.

$$\lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \lim_{\theta\to 0} \frac{1-\cos\theta}{\theta^2}\cdot\frac{1+\cos\theta}{1+\cos\theta} = \lim_{\theta\to 0}\frac{\sin^2\theta}{\theta^2(1+\cos\theta)}$$

$$1-\cos^2\theta = \sin^2\theta$$
$$= (\sin\theta)^2$$

$$= \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)^2\cdot\frac{1}{1+\cos\theta} = 1^2\cdot\frac{1}{1+1} = \frac{1}{2}.$$

$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta} = \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}\cdot\theta = \frac{1}{2}\cdot 0 = 0.$$

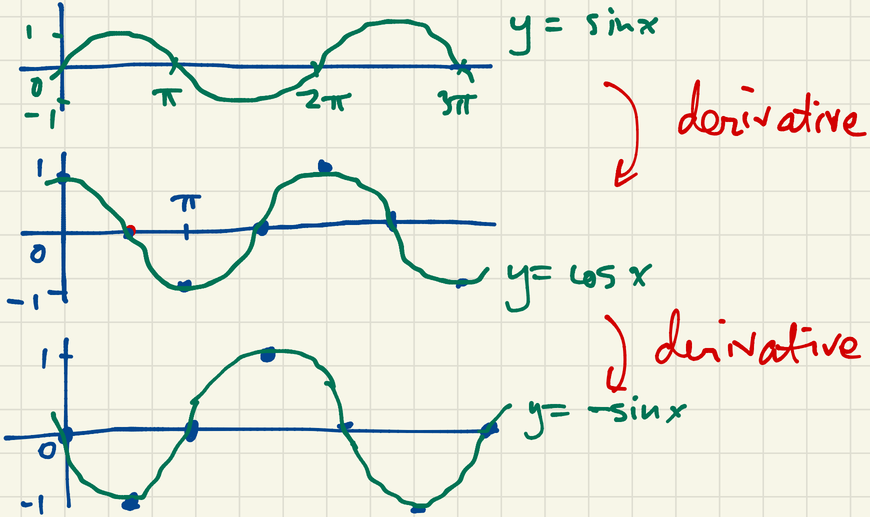
If $f(x) = \sin x$ then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h\to 0} \frac{\sin(x)\cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \left[\cos x \cdot \frac{\sin h}{h} + \sin x \cdot \frac{\cosh - 1}{h} \right] = \cos x \cdot 1 + \sin x \cdot 0 = \cos x$$

$\frac{d}{dx} \sin x = \cos x$

$\frac{d}{dx} \cos x = -\sin x$



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$$\frac{d}{dx}\sin x = \cos x$$

$$\frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

$$\frac{d}{dx}\sec x = \frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x \cdot 0 - 1\cdot(-\sin x)}{\cos^2 x}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x$$

$$\sec x = \frac{1}{\cos x}$$

$$\cot x = \frac{1}{\tan x} = \frac{\cos x}{\sin x}$$

$$\csc x = \frac{1}{\sin x}$$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$ ← derivative of sin at 0 is 1

derivative of tan at 0

$$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x}{x\cos x} = \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x} = 1\cdot 1 = 1.$$

If $f(x) = \sin x$ then $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\sin h}{h} = 1$

If $f(x) = \tan x$ then $f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h} = \lim_{h\to 0}\frac{\tan h}{h} = 1.$

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

$$\lim_{x\to 0}\frac{\sin 5x}{x} = \lim_{u\to 0}\frac{\sin u}{u/5}$$

$u = 5x$

$\frac{u}{5} = x$

$$= \lim_{u\to 0} 5\cdot\frac{\sin u}{u}$$

$$= 5\cdot 1 = 5$$

Note: $\sin 5x \neq 5\sin x$

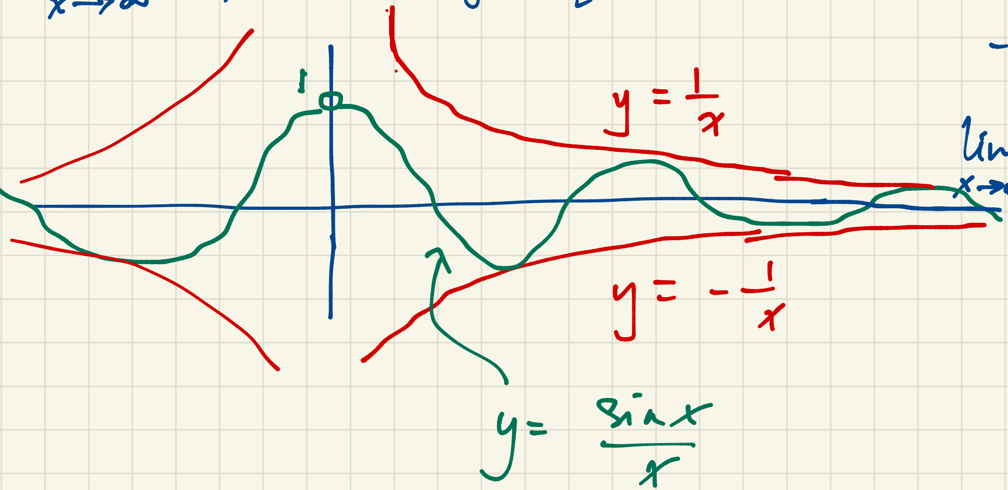
$\sin 2x = 2\sin x\cos x$

$\lim_{x\to\infty}\frac{\sin x}{x} = 0$ by Squeeze Theorem:

$-1 \le \sin x \le 1$

$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$ for $x > 0$

$\lim_{x\to\infty}\frac{1}{x} = 0 = \lim_{x\to\infty}\left(-\frac{1}{x}\right)$



$$\frac{d}{dx}\left(e^x \sin x\right) = e^x \cdot \cos x + e^x \cdot \sin x = e^x\left(\sin x + \cos x\right)$$

$$\frac{d}{dx}\left(x e^x \sin x\right) = x e^x\left(\sin x + \cos x\right) + e^x \sin x = e^x\left(x \sin x + x \cos x + \sin x\right)$$

$$(fg)' = f'g + fg'$$

$$(fgh)' = f'gh + fg'h + fgh'$$

$$\frac{d}{dx}\left(\sin x \cos x\right) = \cos x \cos x + \sin x\left(-\sin x\right) = \cos^2 x - \sin^2 x$$

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$$\frac{d}{dx}\,\frac{\tan x}{e^x + x} = \frac{(e^x + x)\sec^2 x - (\tan x)(e^x + 1)}{(e^x + x)^2}$$

$$\frac{d}{dx}\left(\sin^2 x + \cos^2 x\right) = 2 \sin x \cos x - 2 \sin x \cos x = 0$$

Better: $\frac{d}{dx}\left(\sin^2 x + \cos^2 x\right) = \frac{d}{dx} 1 = 0.$

$$\frac{d}{dx}\left(\sin^2 x\right) = \frac{d}{dx}\left(\sin x \sin x\right)$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2 \sin x \cos x$$

$$\frac{d}{dx}\left(\cos^2 x\right) = \frac{d}{dx}\left(\cos x \cos x\right)$$

$$= -\sin x \cos x + \cos x(-\sin x)$$

$$= -2 \sin x \cos x$$

$$\frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx}\left(e^x + x\right) = e^x + 1$$

A derivative is an instantaneous rate of change.

eg. if $s = s(t)$ is position at time t then

$$\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$ = average velocity during the time interval $t_1 \le t \le t_2$

= average rate of change of position . . . - - . . . . .

$v(t) = s'(t)$ = instantaneous rate of change of position with respect to time at time t

$$= \lim_{\Delta t \to 0} \frac{s(t + \Delta t) - s(t)}{\Delta t}$$

= instantaneous velocity at time t

Note: If position s is in feet and time t is in seconds then velocity (average or instantaneous) is in ft/sec.

Acceleration is the rate of change of velocity i.e. $a(t) = v'(t) = s''(t)$ in ft/sec^2

In differential notation $v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$ (dee two s by dee t squared).

Think of dt as an instantaneous replacement for Δt

$dt^2 = (dt)^2$ $(\Delta t)^2$

d^2s $\Delta^2 s$

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eg. p. 187 #24.

What is the derivative of x^2t^3 ?

$\frac{d}{dx}(x^2t^3) = 2xt^3$ (where t is constant)

$\frac{d}{dt}(x^2t^3) = 3x^2t^2$ (where x is constant)

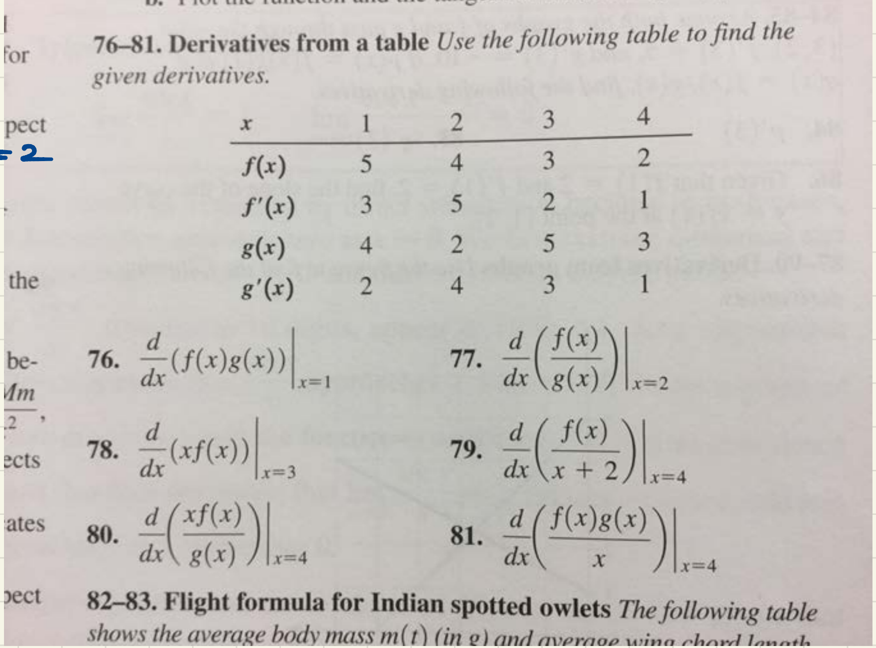
$\left[\frac{d}{dx}(x^2t^3)\right]_{x=3} = \left[2xt^3\right]_{x=3} = 6t^3$ (take derivative, *then* evaluate)

$\frac{d}{dx}(x^2t^3)\Big|_{x=3} = 2xt^3\Big|_{x=3} = 6t^3$

$\frac{d}{dx}\left(x^2t^3\Big|_{x=3}\right) = \frac{d}{dx}(9t^3) = 0$

If $f(x) = x^2t^3$ then $f'(x) = 2xt^3$, $f'(3) = 6t^3$.

If $g(t) = x^2t^3$ then $g'(t) = 3x^2t^2$, $g'(3) = 27x^2$.



76–81. Derivatives from a table *Use the following table to find the given derivatives.*

x	1	2	3	4
$f(x)$	5	4	3	2
$f'(x)$	3	5	2	1
$g(x)$	4	2	5	3
$g'(x)$	2	4	3	1

76. $\frac{d}{dx}(f(x)g(x))\Big|_{x=1}$

77. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=2}$

78. $\frac{d}{dx}(xf(x))\Big|_{x=3}$

79. $\frac{d}{dx}\left(\frac{f(x)}{x+2}\right)\Big|_{x=4}$

80. $\frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right)\Big|_{x=4}$

81. $\frac{d}{dx}\left(\frac{f(x)g(x)}{x}\right)\Big|_{x=4}$

82–83. Flight formula for Indian spotted owlets *The following table shows the average body mass m(t) (in g) and average wing chord length*

$$\frac{d}{dx}\frac{f(x)}{g(x)}\Big|_{x=2} = \frac{g(x)f'(x)-f(x)g'(x)}{g(x)^2}\Big|_{x=2}$$

$$= \frac{g(2)f'(2)-f(2)g'(2)}{g(2)^2}$$

$$= \frac{2\cdot5-4\cdot4}{2^2} = \frac{-6}{4} = -\frac{3}{2}$$

OR

$$\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2)-f(2)g'(2)}{g(2)^2}$$

$$= \frac{2\cdot5-4\cdot4}{2^2} = \frac{-6}{4} = -\frac{3}{2}$$

#76. $(fg)'(1) = f'(1)g(1) + f(1)g'(1) = 3\cdot4 + 5\cdot2 = 22$

Given $y = x^3 - 3x$, find y'. Answer: $y' = 3x^2 - 3$. (Here we safely assume $y' = \frac{dy}{dx}$.)

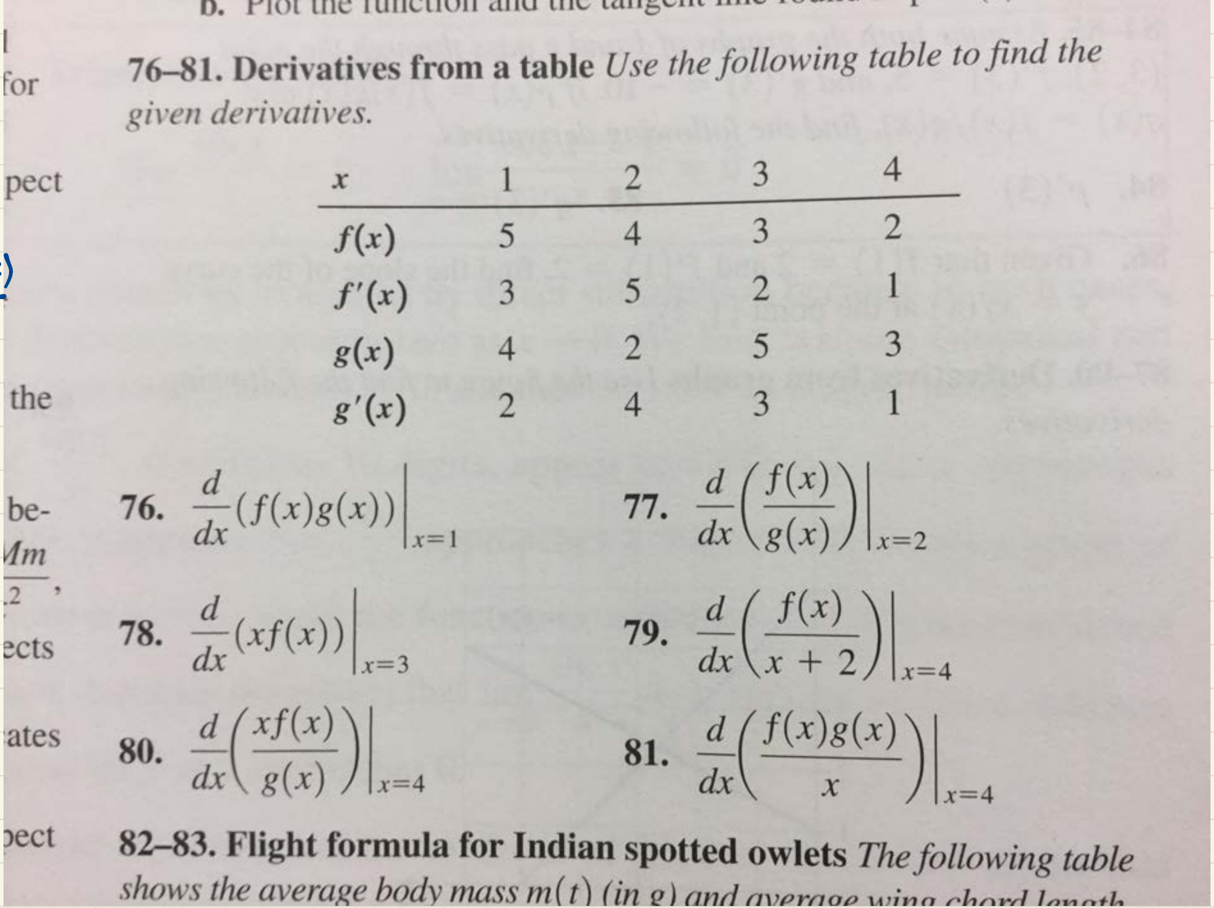
#80. $\frac{d}{dx}\left(\frac{xf(x)}{g(x)}\right)\Big|_{x=4}$

$= \frac{g(x)(f(x)+xf'(x)) - xf(x)g'(x)}{g(x)^2}\Big|_{x=4}$

$= \frac{g(4)(f(4)+4f'(4)) - 4f(4)g'(4)}{g(4)^2}$

$= \frac{3(2+4\cdot1) - 4\cdot2\cdot1}{3^2}$

$= \frac{10}{9}$



b. Plot the function and the tangent

76–81. Derivatives from a table *Use the following table to find the given derivatives.*

x	1	2	3	4
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$g'(x)$	2	4	3	1

76. $\frac{d}{dx}(f(x)g(x))\Big|_{x=1}$

77. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\Big|_{x=2}$

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81. $\frac{d}{dx}\left(\frac{f(x)g(x)}{x}\right)\Big|_{x=4}$

82–83. Flight formula for Indian spotted owlets *The following table shows the average body mass $m(t)$ (in g) and average wing chord length*