Math 2200-01 (Calculus I) Spring 2020

Book 1

Jan 27

Calculus I: Single-variable calculus $y = f(x)$ for example (one input variable x, one output variable). Derivatives (rates of change): <u>differential</u> calculus.

Calculus II: also single-variable. <u>Integral</u> calculus.

Calculus III: multivariable ie. several input variables and/or several output variables

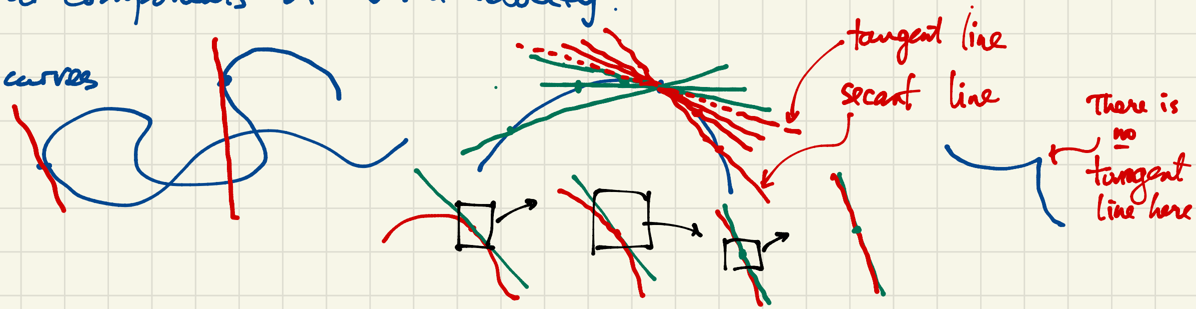
<u>eg</u>. position $(x(t), y(t), z(t))$ of an object at time t: one input t, three output variables $x(t), y(t), z(t)$.

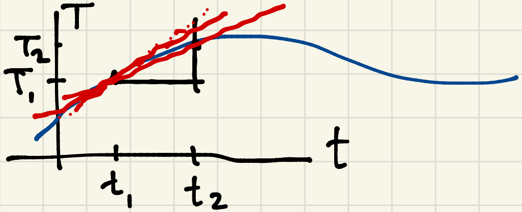
<u>Eg</u>. Temperature in this room as a function of position $T(x,y,z)$ (three inputs x,y,z; one output T)

<u>Eg</u>. Wind velocity as a function of position: three inputs x,y,z; three outputs are the components of wind velocity.

Jan 28

Tangent lines to curves





Temperature T as a function of time t

During the time interval $[t_1, t_2]$ i.e. $t_1 \le t \le t_2$ the temperature rises from T_1 to T_2.

The average rate of change of temperature during this time interval is

$$\frac{\Delta T}{\Delta t} = \frac{T_2 - T_1}{t_2 - t_1}$$

$T_2 - T_1 \leftarrow$ change in temperature

$t_2 - t_1 \leftarrow$ time elapsed.

= slope of the secant line from (t_1, T_1) to (t_2, T_2) on the graph.

We want to understand the instantaneous rate of change of temperature at time t_1. To determine this, first consider the average rate of change over smaller and smaller time intervals $[t_1, t_2]$ where we take $t_2 \to t_1$ (t_2 gets closer and closer to t_1).

Jan 29

Eg. In my example, $t_1 = 3$.

t_2	$\frac{T_2 - T_1}{t_2 - t_1}$	
4	2	degrees/hour
3.2	2.17	
3.1	2.19	
3.001	2.197	
2.9	2.204	
2.7	2.23	
2	2.31	

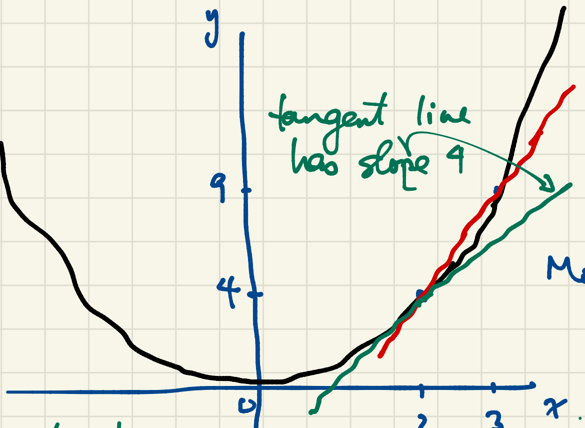
The limit is 2.2.

(The temperature at 3pm is changing at a rate of 2.2 degrees per hour.

We write $\lim\limits_{t_2 \to 3} \frac{T_2 - T_1}{t_2 - t_1} = 2.2$

(the limit of $\frac{T_2 - T_1}{t_2 - t_1}$ is 2.2 as t_2 approaches 3).

A second example using a polynomial function $y = f(x) = x^2$. Find the rate of change of y with respect to x at $x=2$.



The secant line joining the points $(2,4)$ and $(3,9)$ on the curve has slope

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5.$$

$f(2)=4$

More generally, if we join the points $(2,4)$ and $(x, f(x))$ on the curve, the secant line has slope

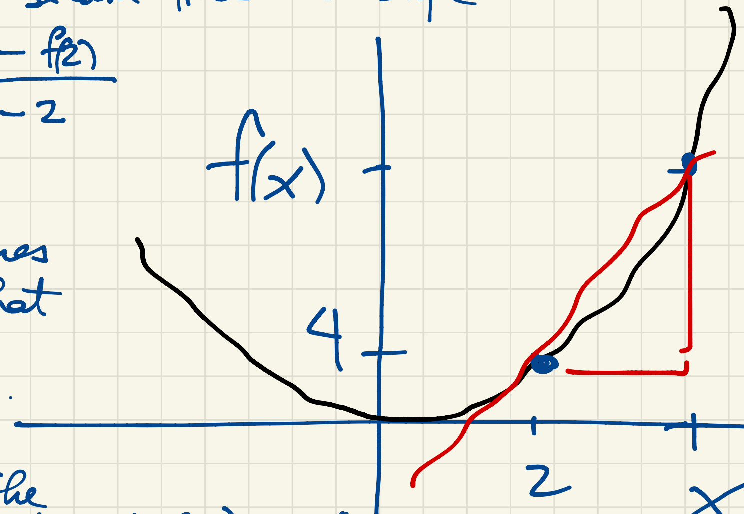
$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(2)}{x-2}$$

The tangent line at $(2,4)$ is
$y-4 = 4(x-2)$ i.e.
$y = 4x-4$.

x	$\dfrac{f(x)-f(2)}{x-2}$
3	5
2.5	4.5
2.1	4.1
2.01	4.01
1.99	3.99
1.9	3.9
1.5	3.5
1	3

Based on the table of values we guess that

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = 4.$$



i.e. the slope of the tangent line to the graph at $(2,4)$ is 4.

If a function has a sufficiently nice formula eg. polynomial, then we have algebraic rules that provide definite ways to evaluate limits, eliminating guesswork based on the graph or table of values.

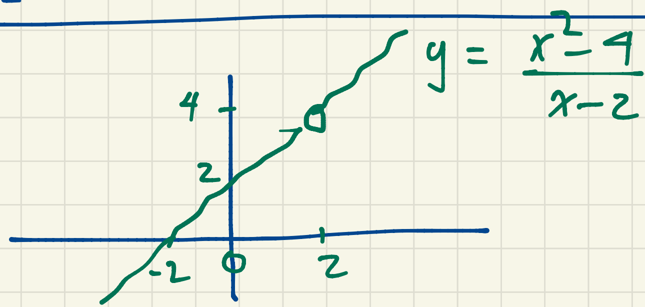
<u>Eg.</u> Find the slope of the tangent line to the graph of $y = x^2$ at $(2,4)$.

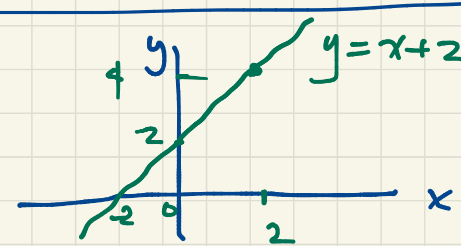
<u>Solution</u> The secant line from $(2,4)$ to $(x, f(x)) = (x, x^2)$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2 \quad \text{for } x \neq 2.$$

The slope of the tangent line is

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 2+2 = 4.$$





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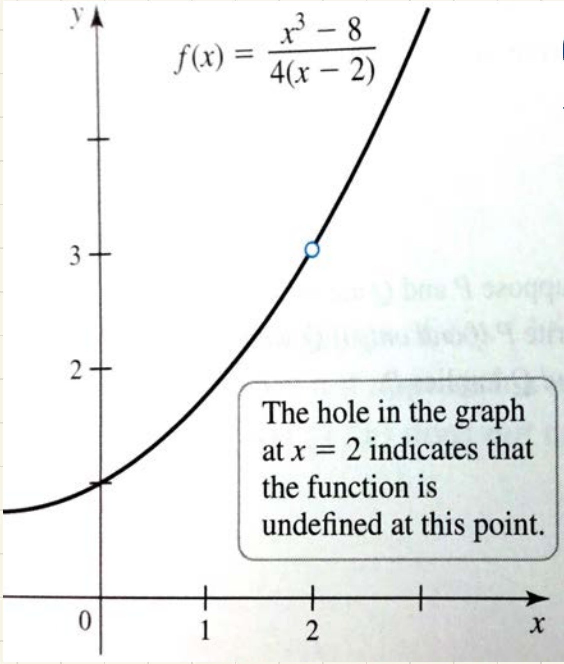
Both of these functions satisfy $\lim_{x\to 2} f(x) = 4$

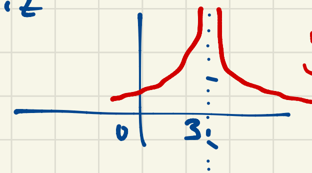


$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 8}{4(x-2)} = 3$$

Feb 10

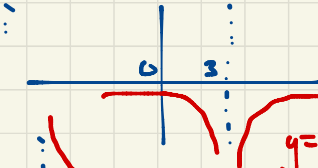
Compare: Friday's quiz

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$$



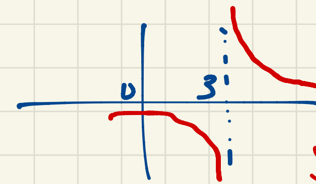
$y = \dfrac{1}{(x-3)^2}$

$$\lim_{x \to 3} \frac{-1}{(x-3)^2} = -\infty$$



$y = \dfrac{-1}{(x-3)^2}$

$\lim_{x \to 3} \dfrac{1}{x-3}$ does not exist



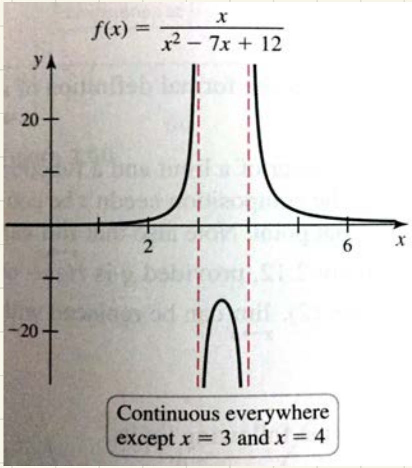
$y = \dfrac{1}{x-3}$

$$\lim_{x \to 3^+} \frac{1}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{1}{x-3} = -\infty$$

$$\lim_{x \to \infty} \frac{1}{x-3} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x-3} = 0$$

Sec 2.6



Continuous everywhere except $x = 3$ and $x = 4$

A function f is continuous at a if $\lim_{x \to a} f(x) = f(a)$.

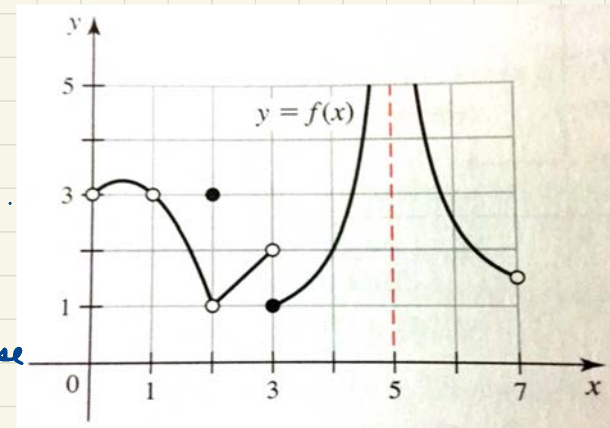
Explicitly, this requires that

(i) f must be defined at a, ie. $f(a)$ exists;

(ii) f must have a limit at a; and

(iii) the values in (i) and (ii) must agree.

Eg. for the function f on the right,



- f is discontinuous at 5;
- f " " " 3.

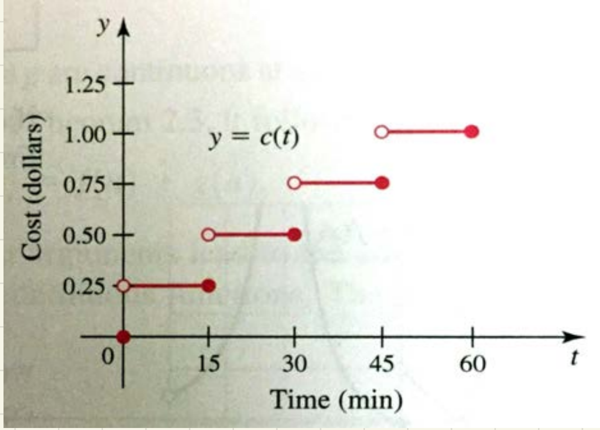
  $f(3) = 1$, $\lim_{x \to 3} f(x)$ does not exist.

- f is not continuous at 2.

  $f(2) = 3$, $\lim_{x \to 2} f(x) = 1$ but these two values do not agree!

- f is discontinuous at 1. Although $\lim_{x \to 1} f(x) = 3$, f is not defined at 1.

f is continuous on $(0, 7)$ ie. $0 < x < 7$ except at 1, 2, 3, 5.



Eg. the cost of parking at a meter is 25¢ for each 15 minutes. The cost $c(t)$ as a function of time is discontinuous at $t = 0, 15, 30, 45, 60, \cdots$ At each of these points of discontinuity, c is left-continuous (i.e. $\lim_{t \to a^-} f(t) = f(a)$) but not right-continuous (i.e. $\lim_{t \to a^+} f(t) \neq f(a)$).

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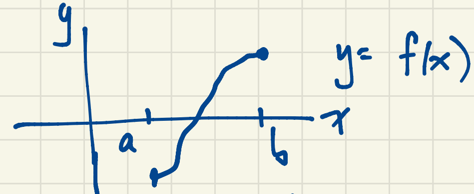
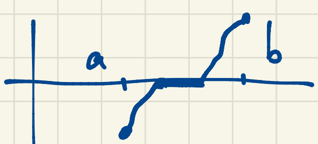
Why do we care about continuity?



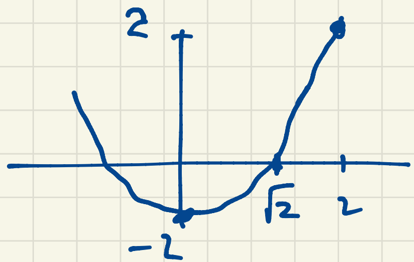
If f is continuous with $f(a) < 0$ and $f(b) > 0$ then there exists c, $a < c < b$, such that $f(c) = 0$. (Intermediate Value Theorem)

Remarks: The point c might not be unique i.e. there might be more than one c with this property.

or

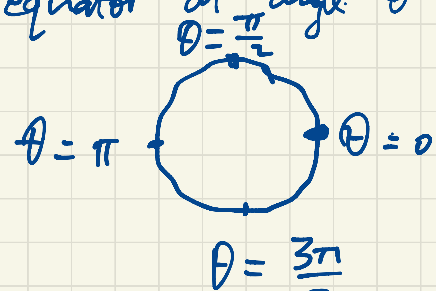


What is $\sqrt{2}$? Why does such a number exist? Consider $f(x) = x^2 - 2$.
f is continuous because it is a polynomial (See Sec 2.6). By the Intermediate



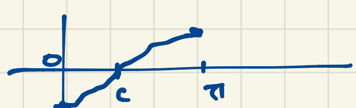
Value theorem (since $f(0) < 0$, $f(2) > 0$) there exists c between 0 and 2 such that $f(c) = 0$.
Later, as we'll see, there is only one such c.
We call this value $\sqrt{2}$.

Another example: At this moment there are two points which are antipodes on the Earth's surface having exactly the same temperature.
Consider the equator and let $T(\theta)$, $0 \le \theta < 2\pi$, be the temperature on the equator at angle θ with respect to $0°$ longitude (i.e. θ is longitude).

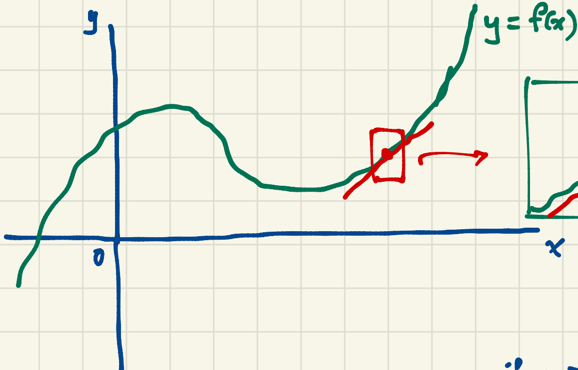


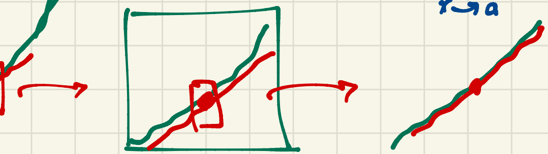
$f(\theta) = T(\theta + \pi) - T(\theta)$ = difference in temperature between longitude θ and its antipode (at $\theta + \pi$).

If $f(0) < 0$ i.e. $T(\pi) < T(0)$ then $f(\pi) > 0$.



There exists c, $0 < c < \pi$ such that $f(c) = 0$. i.e. $T(c) = T(c + \pi)$.





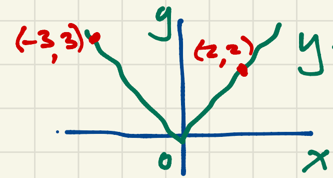
$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$(h = \Delta x = x - a)$

$\Delta y = f(x) - f(a) = f(a+h) - f(a)$

Feb 14 ♡

Eg. $g(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$



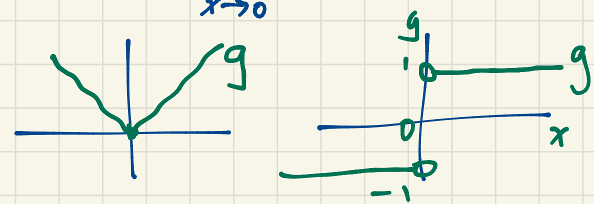
$$g'(2) = \lim_{x \to 2} \frac{g(x) - g(2)}{x - 2} = \lim_{x \to 2} \frac{|x| - 2}{x - 2} = \lim_{x \to 2} \frac{x - 2}{x - 2} = \lim_{x \to 2} 1 = 1.$$

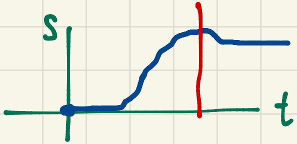
$$g'(-3) = \lim_{x \to -3} \frac{|x| - 3}{x + 3} = \lim_{x \to -3} \frac{-x - 3}{x + 3} = \lim_{x \to -3} (-1) = -1.$$

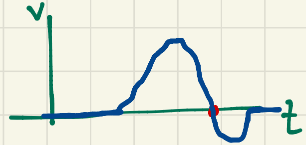
$g'(0) = \lim_{x \to 0} \frac{|x| - 0}{x - 0} = \lim_{x \to 0} \frac{|x|}{x}$ does not exist $\left(\lim_{x \to 0^+} \frac{|x|}{x} = 1 \text{ whereas } \lim_{x \to 0^-} \frac{|x|}{x} = -1 \right)$.

$g'(0)$ does not exist. $|x|$ is not differentiable at 0.

$g'(a) = \begin{cases} 1 & \text{if } a > 0; \\ -1 & \text{if } a < 0. \\ & (\text{undefined if } a = 0). \end{cases}$







$s =$ position (displacement)

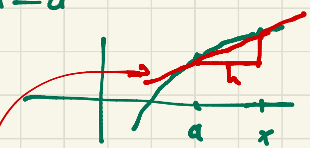
$t =$ time

$v =$ velocity $\qquad v(t) = s'(t)$

Feb 17

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$h = \Delta x = x - a$



secant line from $(a, f(a))$ to $(x, f(x))$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - a}{x - a}$

$$= \frac{f(a+h) - f(a)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Eg. $f(x) = x^2$

$f(3) = 9$

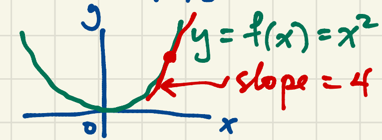
$f(-2) = 4$

$f(w) = w^2$

$f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$

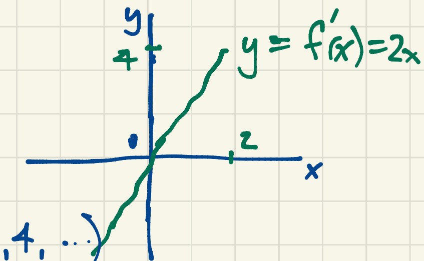
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} \frac{(2x+h)h}{h} = \lim_{h\to 0} (2x+h) = 2x.$$



The derivative of $f(x) = x^2$ with respect to x is $f'(x) = 2x$.

$$\boxed{\frac{d\,x^2}{dx} = 2x}$$



More generally, let n be a positive integer (i.e. n is one of $1, 2, 3, 4, \ldots$)

Consider the power function $f(x) = x^n$.

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x-a}$$

$$= \lim_{x\to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$

$$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n} = na^{n-1}$$

So $f'(x) = nx^{n-1}$

$$\frac{dx^n}{dx} = nx^{n-1}$$

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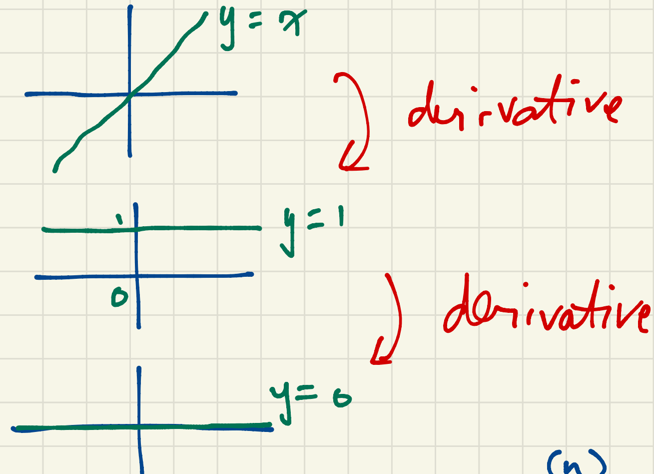
$$\frac{d}{dx}x^2 = 2x \qquad \underbrace{\frac{d}{dx}x^2}_{\frac{dx^2}{dx}}$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{dx}{dx} = 1$$

$$\frac{d1}{dx} = 0$$



$$f \xrightarrow[\text{derivative}]{} f' \xrightarrow[\text{derivative}]{} f'' \xrightarrow[\text{derivative}]{} f''' \longrightarrow \cdots \longrightarrow f^{(n)}$$

n^{th} derivative of f

$f^{(1)} = f' =$ (first) derivative of f

$f^{(2)} = f'' =$ second derivative of f etc.

Eg. if $f(x) = x^3$ then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

If c is constant then $\frac{d}{dx} cy = c \frac{dy}{dx}$ i.e. $(cf)' = cf'$

$(f+g)' = f' + g'$ i.e. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Now we can take the derivative of any polynomial eg.

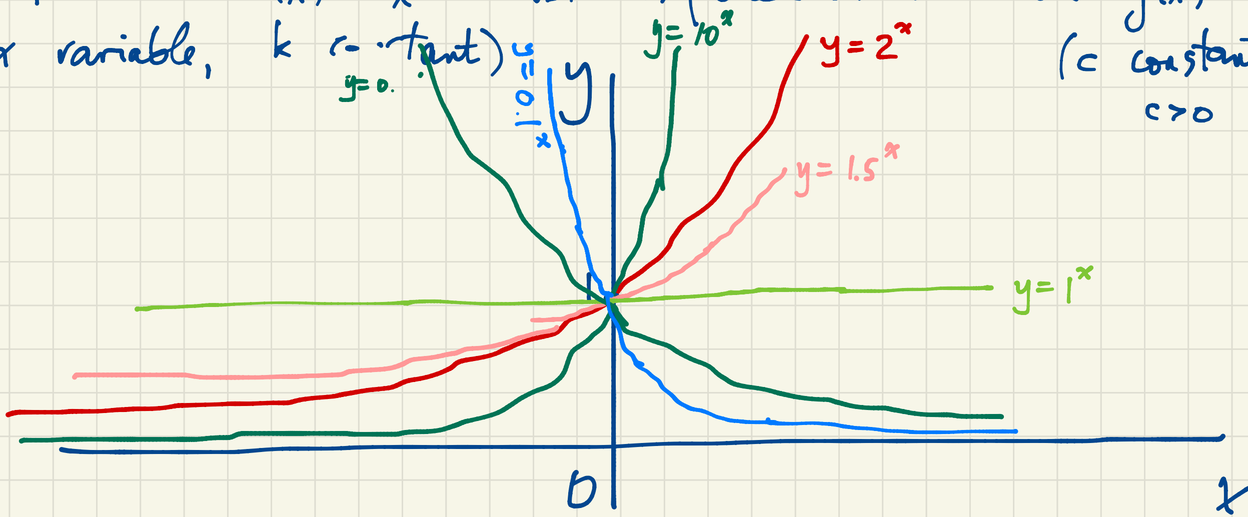
$$\frac{d}{dx}\left(7x^3 - 3x^2 - 5x + 11\right) = 21x^2 - 6x - 5$$

i.e. if $f(x) = 7x^3 - 3x^2 - 5x + 11$ then $f'(x) = 21x^2 - 6x - 5$.

Power function $f(x) = x^k$ vs. Exponential function $g(x) = c^x$

(x variable, k constant) (c constant, x variable)

$c > 0$



The function $f(x) = c^x$ exhibits exponential growth for $c > 1$ (faster growth than any power function) and exponential decay for $0 < c < 1$. As we vary the base c of the exponential, the curve $y = c^x$ passes through the intercept $(0,1)$ with varying slope, e.g. slope $\simeq 0.693$ when $c = 2$ and slope $\simeq 1.099$ when $c = 3$. We expect that for some c between 2 and 3, the curve $y = 2^x$ will pass through $(1,0)$ with slope exactly 1 (and this expectation can be justified using the Intermediate Value Theorem). Accordingly, we define e to be the unique constant for which the curve $y = e^x$ has tangent line of slope exactly 1 at the point $(0,1)$. Thus by definition, if $f(x) = e^x$, then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and e is the unique number with this property. (So we may take this as our definition of e.) It may be shown that $e \simeq 2.71828\ldots$.

If $f(x) = e^x$ then $f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Feb 19

$$e^{a(x+h)} = e^{ax+ah} = e^{ax} \cdot e^{ah}$$

If $f(x) = e^{ax}$ (a constant) then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{e^{a(x+h)} - e^{ax}}{h} = \lim_{h\to 0} \frac{e^{ax}e^{ah} - e^{ax}}{h} = \lim_{h\to 0} e^{ax} \cdot \frac{e^{ah} - 1}{h}$$

$$= e^{ax} \lim_{h\to 0} \frac{e^{ah} - 1}{h} = e^{ax} \lim_{t\to 0} \frac{e^t - 1}{t/a} = a e^{ax} \lim_{t \to 0} \frac{e^t - 1}{t} = a e^{ax} \cdot 1 = a e^{ax}.$$

Substitute $t = ah$

i.e. $h = \frac{t}{a}$

$$\boxed{\frac{d}{dx} e^{ax} = a e^{ax}}$$

Set $a = \ln 2 \iff e^a = 2$.

$$\iff (e^a)^x = 2^x$$

$$\iff e^{ax} = 2^x$$

Exponentials and logarithms are inverse functions

$$a \underset{\ln}{\overset{\exp}{\rightleftarrows}} 2$$

$$2^x = e^{(\ln 2)x}$$

$$\frac{d}{dx} 2^x = (\ln 2)\, 2^x$$

$$\frac{d}{dx} c^x = c^x \cdot \ln c$$

Question: For which functions f does $f' = f$? i.e. $\frac{dy}{dx} = y$?

Answer: Functions of the form $f(x) = ke^x$ where k is constant have $f'(x) = ke^x = f(x)$.
It turns out (later...) that these are the only solutions.

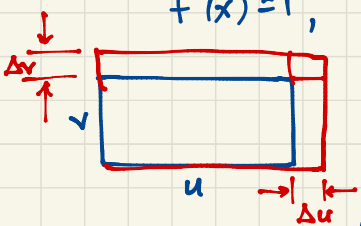
$$\frac{d}{dx} e^{4x+7} = \frac{d}{dx} e^7 \cdot e^{4x} = e^7 \cdot 4e^{4x} = 4e^{4x+7}.$$

$$a^{mn} = (a^m)^n$$
$$a^{m+n} = a^m \cdot a^n$$

Sec 3.4: Product Rule. $(f+g)' = f' + g'$ but $(fg)' \neq f'g'$.

Eg. $f(x) = x+3$, $g(x) = x^2$. $\quad f(x)g(x) = (x+3)x^2 = x^3 + 3x^2$

$f'(x) = 1$, $g'(x) = 2x$ $\quad (fg)'(x) = 3x^2 + 6x \neq f'(x)g'(x)$



Original area $A = uv$ at time t

New area $= (u+\Delta u)(v+\Delta v) = uv + u\Delta v + v\Delta u + (\Delta u)(\Delta v)$ at time $t + \Delta t$

Imagine a rectangle $u \times v$ which grows in time. If we increase length u by Δu and the width v by Δv then what is the change in area?

Feb 21

$$\frac{\Delta A}{\Delta t} = \frac{u\Delta v + v\Delta u + \Delta u \Delta v}{\Delta t} = u\frac{\Delta v}{\Delta t} + \frac{\Delta u}{\Delta t}v + \underbrace{\Delta u}_{\downarrow\, 0} \cdot \underbrace{\frac{\Delta v}{\Delta t}}_{\downarrow\, \frac{dv}{dt}}.$$

Let $\Delta t \to 0$. In the limit,

$$\frac{dA}{dt} = u\frac{dv}{dt} + \frac{du}{dt}v + 0$$

$$\frac{d}{dt}(uv) = u\frac{dv}{dt} + \frac{du}{dt}v \quad \text{(the product rule)}$$

$$(fg)' = f'g + fg'$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dt} - u\frac{dv}{dt}}{v^2} \quad \text{(quotient rule)}$$

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}$$

$$u = \frac{u}{v}\cdot v$$

$$\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{v}\cdot v\right) = \frac{d}{dt}\left(\frac{u}{v}\right)\cdot v + \frac{u}{v}\cdot\frac{dv}{dt}$$

$$\frac{1}{v}\frac{du}{dt} = \frac{d}{dt}\left(\frac{u}{v}\right) + \frac{u}{v^2}\frac{dv}{dt}$$

$$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{1}{v}\frac{du}{dt} - \frac{u}{v^2}\frac{dv}{dt}$$

$$\frac{d}{dx}(xe^x) = x\cdot e^x + e^x\cdot 1 = (x+1)e^x$$

slow motion:

$$\frac{d}{dx}(xe^x) = x\frac{de^x}{dx} + \frac{dx}{dx}e^x = xe^x + 1\cdot e^x = (x+1)e^x.$$

$$\frac{d}{dx}(x+3)x^2 = (x+3)\cdot 2x + 1\cdot x^2 = 3x^2 + 6x$$

$$\frac{d}{dx}(xe^{-x}) = 1\cdot e^{-x} + x(-e^{-x}) = e^{-x} - xe^{-x} = (1-x)e^{-x}$$

$$\frac{d}{dx}e^{ax} = ae^{ax}$$

Alternatively,

$$\frac{d}{dx}(xe^{-x}) = \frac{d}{dx}\left(\frac{x}{e^x}\right) = \frac{e^x\cdot 1 - x\cdot e^x}{(e^x)^2} = \frac{1-x}{e^x} = (1-x)e^{-x}.$$

Eg. $$\frac{d}{dx}\frac{4x^2-3}{(x+1)^2} = \frac{(x+1)^2\cdot 8x - (4x^2-3)\cdot(2x+2)}{(x+1)^4}$$

$$= \frac{(x+1)\cdot 8x - (4x^2-3)\cdot 2}{(x+1)^3} = \frac{8x+6}{(x+1)^3}$$

$$\frac{d}{dx}(x+1)^2 = \frac{d}{dx}(x^2+2x+1)$$

$$= 2x+2 = 2(x+1)$$