# Math 2200-01 (Calculus I) Spring 2020

Book 1

Jan 27

Calculus I: Single-variable calculus $y = f(x)$ for example (one input variable $x$, one output variable). Derivatives (rates of change): differential calculus.

Calculus II: also single-variable. Integral calculus.

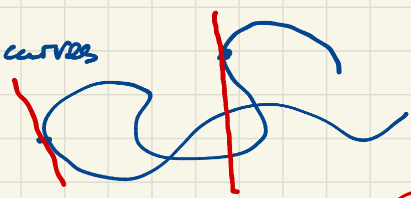
Calculus III: multivariable ie. several input variables and/or several output variables
eg. position $(x(t), y(t), z(t))$ of an object at time $t$: one input $t$, three output variables $x(t), y(t), z(t)$.

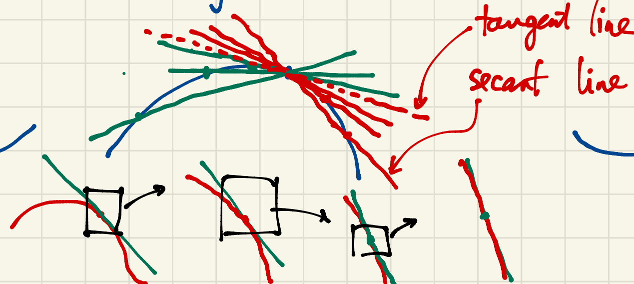
Eg. Temperature in this room as a function of position $T(x,y,z)$ (three inputs $x,y,z$; one output $T$)

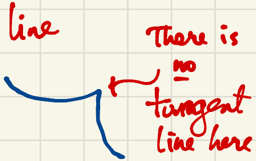
Eg. Wind velocity as a function of position: three inputs $x,y,z$; three outputs are the components of wind velocity.

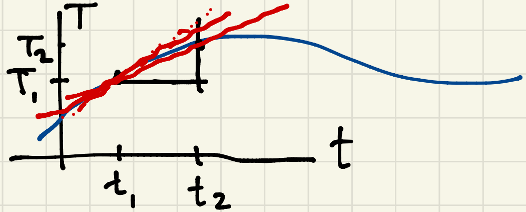
Jan 28

Tangent lines to curves







Temperature $T$ as a function of time $t$

During the time interval $[t_1, t_2]$ i.e. $t_1 \le t \le t_2$ the temperature rises from $T_1$ to $T_2$.

The average rate of change of temperature during this time interval is

$$\frac{\Delta T}{\Delta t} = \frac{T_2 - T_1}{t_2 - t_1}$$

$T_2 - T_1 \leftarrow$ change in temperature

$t_2 - t_1 \leftarrow$ time elapsed.

= slope of the secant line from $(t_1, T_1)$ to $(t_2, T_2)$ on the graph.

We want to understand the instantaneous rate of change of temperature at time $t_1$. To determine this, first consider the average rate of change over smaller and smaller time intervals $[t_1, t_2]$ where we take $t_2 \to t_1$ ($t_2$ gets closer and closer to $t_1$).

Jan 29

Eg.

In my example, $t_1 = 3$.

| $t_2$ | $\frac{T_2 - T_1}{t_2 - t_1}$ |
|---|---|
| 4 | 2 degrees/hour |
| 3.2 | 2.17 |
| 3.1 | 2.19 |
| 3.001 | 2.197 |
| 2.9 | 2.204 |
| 2.7 | 2.23 |
| 2 | 2.31 |

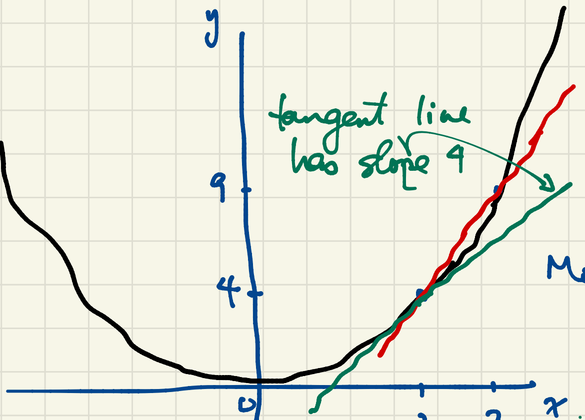
The limit is 2.2.

(The temperature at 3pm is changing at a rate of 2.2 degrees per hour.

We write $\lim\limits_{t_2 \to 3} \frac{T_2 - T_1}{t_2 - t_1} = 2.2$

(the limit of $\frac{T_2 - T_1}{t_2 - t_1}$ is 2.2 as $t_2$ approaches 3).

A second example using a polynomial function $y = f(x) = x^2$. Find the rate of change of $y$ with respect to $x$ at $x=2$.




The secant line joining the points $(2,4)$ and $(3,9)$ on the curve has slope

$$\frac{\Delta y}{\Delta x} = \frac{9-4}{3-2} = 5.$$

More generally, if we join the points $(2,4)$ and $(x, f(x))$ on the curve, the secant line has slope

$f(2)=4$

$$\frac{\Delta y}{\Delta x} = \frac{f(x)-f(2)}{x-2}$$

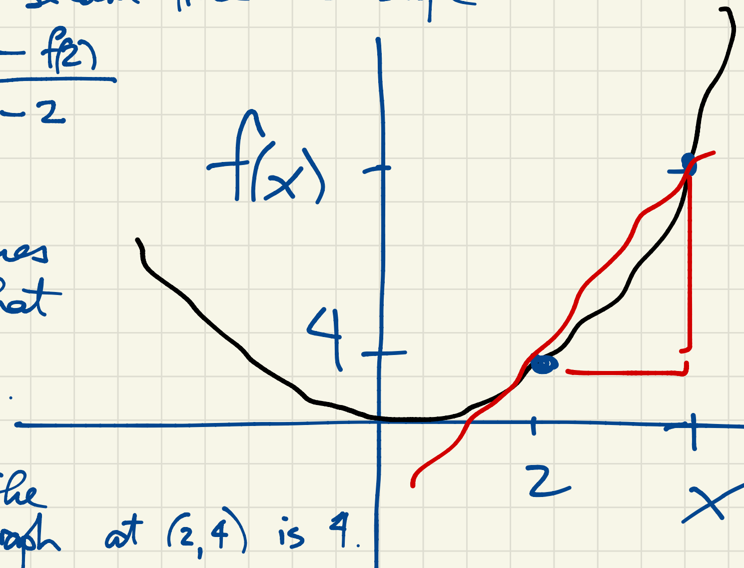
The tangent line at $(2,4)$ is $y-4 = 4(x-2)$ i.e. $y = 4x-4$.

| $x$ | $\frac{f(x)-f(2)}{x-2}$ |
|---|---|
| 3 | 5 |
| 2.5 | 4.5 |
| 2.1 | 4.1 |
| 2.01 | 4.01 |
| 1.99 | 3.99 |
| 1.9 | 3.9 |
| 1.5 | 3.5 |
| 1 | 3 |

Based on the table of values we guess that

$$\lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = 4.$$

i.e. the slope of the tangent line to the graph at $(2,4)$ is 4.



If a function has a sufficiently nice formula eg. polynomial, then we have algebraic rules that provide definite ways to evaluate limits, eliminating guesswork based on the graph or table of values.

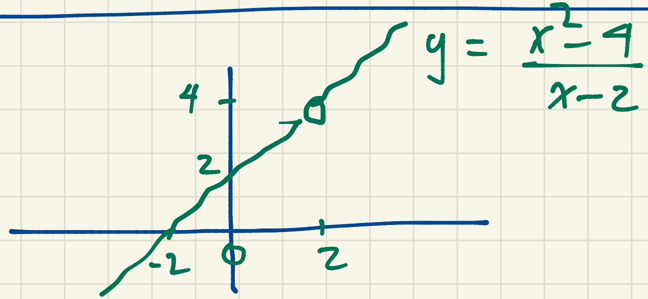
Eg. Find the slope of the tangent line to the graph of $y=x^2$ at $(2,4)$.

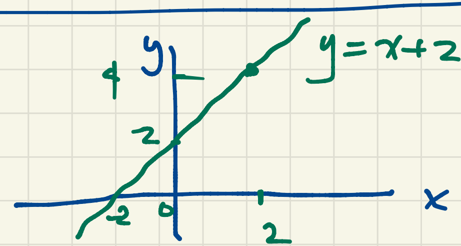
Solution The secant line from $(2,4)$ to $(x, f(x)) = (x, x^2)$ has slope

$$\frac{\Delta y}{\Delta x} = \frac{x^2-4}{x-2} = \frac{(x+2)(x-2)}{x-2} = x+2 \quad \text{for } x \neq 2.$$

The slope of the tangent line is

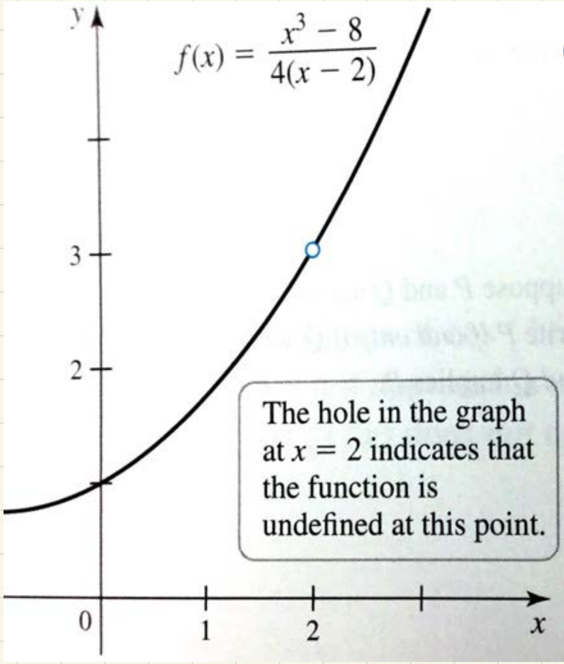
$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} (x+2) = 2+2=4.$$




Jan 31

Both of these functions satisfy $\lim_{x\to 2} f(x) = 4$

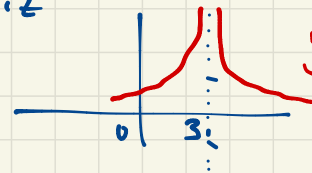



$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 8}{4(x-2)} = 3$$

Feb 10

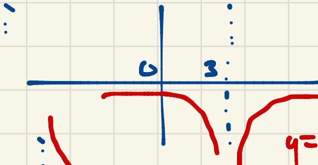
Compare: Friday's quiz

$$\lim_{x \to 3} \frac{1}{(x-3)^2} = \infty$$



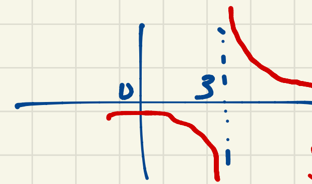

$y = \frac{1}{(x-3)^2}$

$$\lim_{x \to 3} \frac{-1}{(x-3)^2} = -\infty$$




$y = \frac{-1}{(x-3)^2}$

$\lim_{x \to 3} \frac{1}{x-3}$ does not exist




$y = \frac{1}{x-3}$

$$\lim_{x \to 3^+} \frac{1}{x-3} = \infty \qquad \lim_{x \to 3^-} \frac{1}{x-3} = -\infty$$

$$\lim_{x \to \infty} \frac{1}{x-3} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x-3} = 0$$

# Sec 2.6

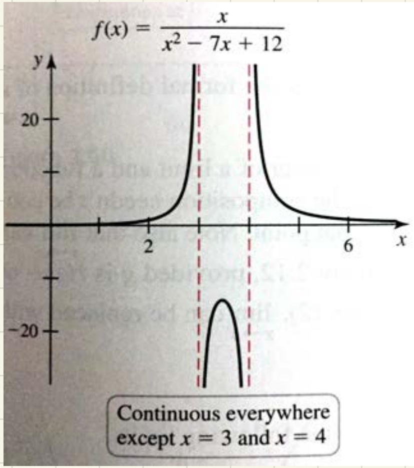
A function $f$ is continuous at $a$ if $\lim_{x\to a} f(x) = f(a)$.

Explicitly, this requires that

(i) $f$ must be defined at $a$, ie. $f(a)$ exists;

(ii) $f$ must have a limit at $a$; and

(iii) the values in (i) and (ii) must agree.



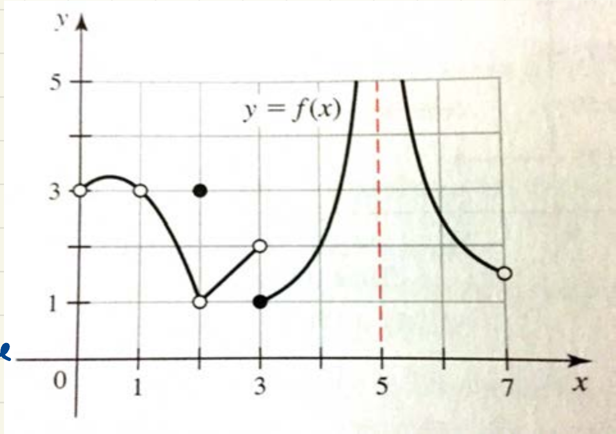

Eg. for the function $f$ on the right,

- $f$ is discontinuous at 5;
- $f$ " " " 3.

  $f(3) = 1$, $\lim_{x\to 3} f(x)$ does not exist.

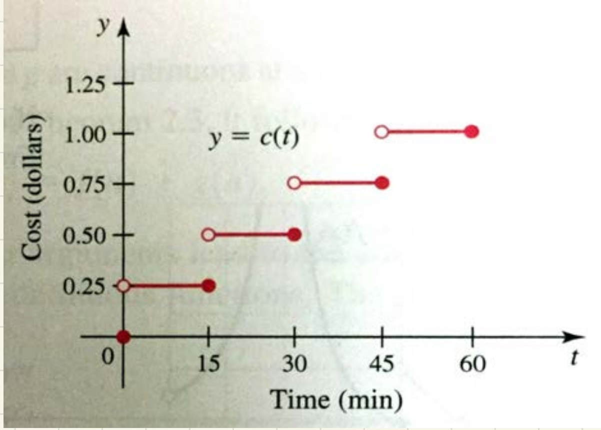
- $f$ is not continuous at 2.

  $f(2) = 3$, $\lim_{x\to 2} f(x) = 1$ but these two values do not agree!




- $f$ is discontinuous at 1. Although $\lim_{x\to 1} f(x) = 3$, $f$ is not defined at 1.

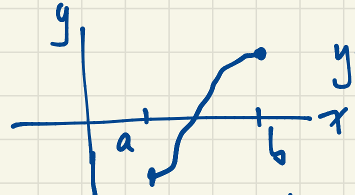
$f$ is continuous on $(0, 7)$ ie. $0 < x < 7$ except at 1, 2, 3, 5.




Eg. the cost of parking at a meter is 25¢ for each 15 minutes. The cost $c(t)$ as a function of time is discontinuous at $t = 0, 15, 30, 45, 60, \cdots$ At each of these points of discontinuity, $c$ is left-continuous (i.e. $\lim_{t \to a^-} f(t) = f(a)$) but not right-continuous (i.e. $\lim_{t \to a^+} f(t) \neq f(a)$).

Feb 11

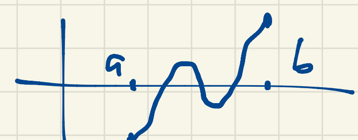
Why do we care about continuity?



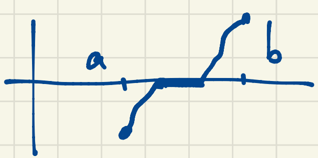

$y = f(x)$

If $f$ is continuous with $f(a) < 0$ and $f(b) > 0$ then there exists $c$, $a < c < b$, such that $f(c) = 0$. (Intermediate Value Theorem)

Remarks: The point $c$ might not be unique i.e. there might be more than one $c$ with this property.




or

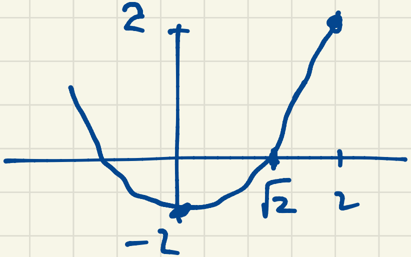


What is $\sqrt{2}$? Why does such a number exist? Consider $f(x) = x^2 - 2$.

$f$ is continuous because it is a polynomial (See Sec 2.6). By the Intermediate Value theorem (since $f(0) < 0$, $f(2) > 0$) there exists $c$ between $0$ and $2$ such that $f(c) = 0$.

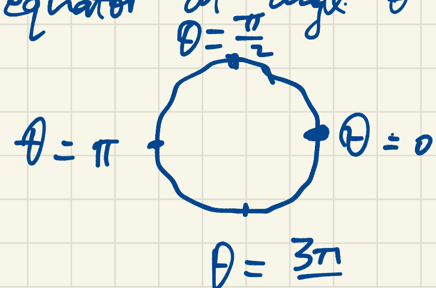
Later, as we'll see, there is only one such $c$.

We call this value $\sqrt{2}$.



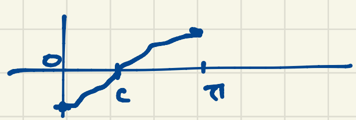

Another example: At this moment there are two points which are antipodes on the Earth's surface having exactly the same temperature.

Consider the equator and let $T(\theta)$, $0 \le \theta < 2\pi$, be the temperature on the equator at angle $\theta$ with respect to $0°$ longitude (i.e. $\theta$ is longitude).

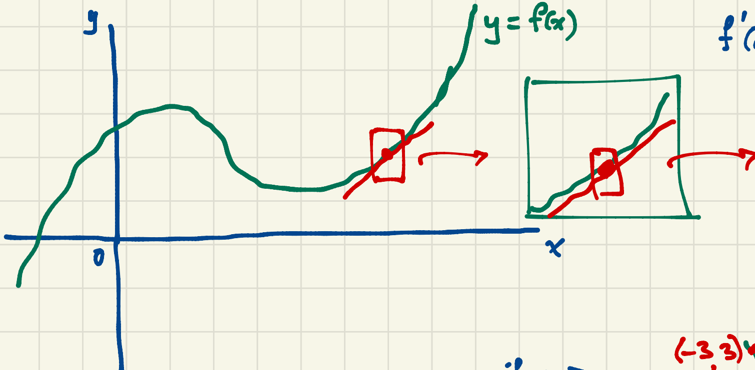


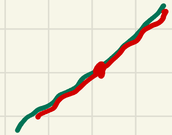

$f(\theta) = T(\theta + \pi) - T(\theta)$ = difference in temperature between longitude $\theta$ and its antipode (at $\theta + \pi$).

If $f(0) < 0$ i.e. $T(\pi) < T(0)$ then $f(\pi) > 0$.




There exists $c$, $0 < c < \pi$ such that $f(c) = 0$. i.e. $T(c) = T(c + \pi)$.



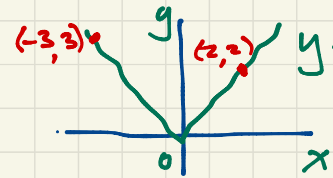
$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$(h = \Delta x = x - a)$

$\Delta y = f(x) - f(a) = f(a+h) - f(a)$

Feb 14 ♡

Eg. $g(x) = |x| = \begin{cases} x, & \text{if } x \geq 0; \\ -x, & \text{if } x < 0. \end{cases}$




$y = g(x) = |x|$

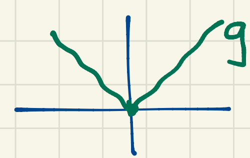
$$g'(2) = \lim_{x\to 2} \frac{g(x)-g(2)}{x-2} = \lim_{x\to 2} \frac{|x|-2}{x-2} = \lim_{x\to 2} \frac{x-2}{x-2} = \lim_{x\to 2} 1 = 1.$$

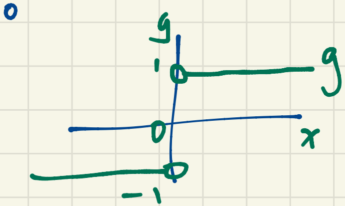
$$g'(-3) = \lim_{x\to -3} \frac{|x|-3}{x+3} = \lim_{x\to -3} \frac{-x-3}{x+3} = \lim_{x\to -3} (-1) = -1.$$

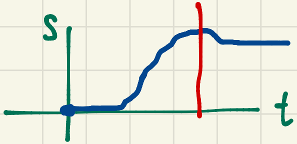
$g'(0) = \lim_{x\to 0} \dfrac{|x|-0}{x-0} = \lim_{x\to 0} \dfrac{|x|}{x}$ does not exist $\left( \lim_{x\to 0^+} \dfrac{|x|}{x} = 1 \text{ whereas } \lim_{x\to 0^-} \dfrac{|x|}{x} = -1 \right)$.

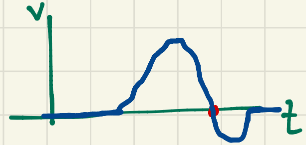
$g'(0)$ does not exist. $|x|$ is not differentiable at $0$.

$g'(a) = \begin{cases} 1 & \text{if } a > 0; \\ -1 & \text{if } a < 0. \\ (\text{undefined if } a = 0). \end{cases}$






Feb 17

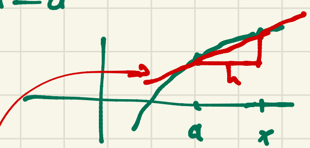
s = position (displacement)

t = time

v = velocity $\qquad v(t) = s'(t)$

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

$h = \Delta x = x - a$




secant line
from $(a, f(a))$ to $(x, f(x))$
has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x)-a}{x-a}$

$$= \frac{f(a+h)-f(a)}{h}$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Eg. $f(x) = x^2$

$f(3) = 9$

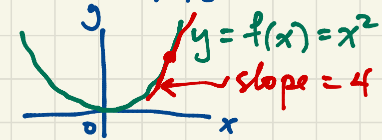
$f(-2) = 4$

$f(w) = w^2$

$f(x+h) = (x+h)^2 = x^2 + 2hx + h^2$

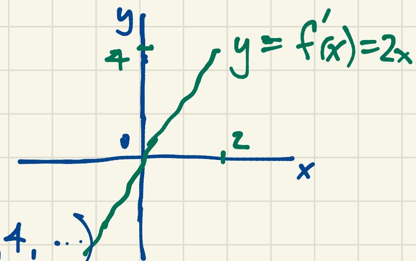
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2hx + h^2 - x^2}{h} = \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} \frac{(2x+h)h}{h} = \lim_{h\to 0} (2x+h) = 2x.$$




The derivative of $f(x) = x^2$ with respect to $x$ is $f'(x) = 2x$.

$$\boxed{\frac{d\,x^2}{dx} = 2x}$$




More generally, let $n$ be a positive integer (i.e. $n$ is one of $1, 2, 3, 4, \ldots$)

Consider the ~~power~~ function $f(x) = x^n$.

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})}{x-a}$$

$$= \lim_{x\to a} (x^{n-1} + ax^{n-2} + a^2x^{n-3} + \cdots + a^{n-2}x + a^{n-1})$$

$$= \underbrace{a^{n-1} + a^{n-1} + a^{n-1} + \cdots + a^{n-1}}_{n} = na^{n-1}$$

So $f'(x) = nx^{n-1}$

$$\frac{dx^n}{dx} = nx^{n-1}$$

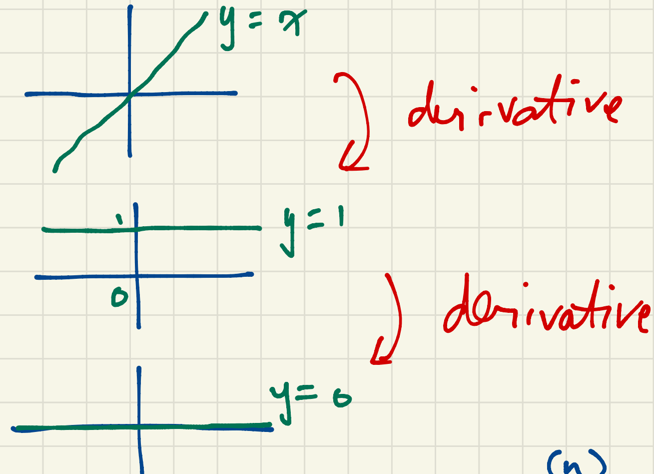
Feb 18

$$\underbrace{\frac{d}{dx}x^2}_{\frac{dx^2}{dx}} = 2x$$

$$\frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{dx}{dx} = 1$$

$$\frac{d1}{dx} = 0$$




$$f \xrightarrow[\text{derivative}]{} f' \xrightarrow[\text{derivative}]{} f'' \xrightarrow[\text{derivative}]{} f''' \longrightarrow \cdots \longrightarrow f^{(n)}$$

$n^{th}$ derivative of $f$

$f^{(1)} = f' =$ (first) derivative of $f$

$f^{(2)} = f'' =$ second derivative of $f$ etc.

Eg. if $f(x) = x^3$ then $f'(x) = 3x^2$, $f''(x) = 6x$, $f'''(x) = 6$, $f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

If $c$ is constant then $\frac{d}{dx} cy = c\frac{dy}{dx}$ i.e. $(cf)' = cf'$

$(f+g)' = f' + g'$ i.e. $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

Now we can take the derivative of any polynomial eg.

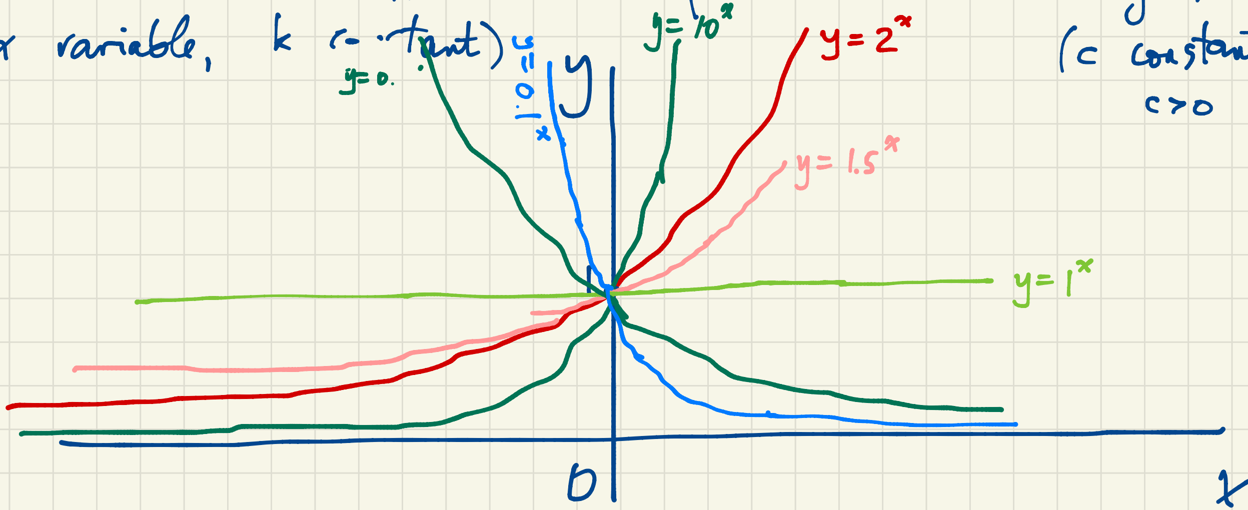
$$\frac{d}{dx}\left(7x^3 - 3x^2 - 5x + 11\right) = 21x^2 - 6x - 5$$

i.e. if $f(x) = 7x^3 - 3x^2 - 5x + 11$ then $f'(x) = 21x^2 - 6x - 5$.

Power function $f(x) = x^k$ vs. Exponential function $g(x) = c^x$

($x$ variable, $k$ constant) ($c$ constant, $x$ variable)

$c > 0$



The function $f(x) = c^x$ exhibits exponential growth for $c > 1$ (faster growth than any power function) and exponential decay for $0 < c < 1$. As we vary the base $c$ of the exponential, the curve $y = c^x$ passes through the intercept $(0,1)$ with varying slope, e.g. slope $\approx 0.693$ when $c = 2$ and slope $\approx 1.099$ when $c = 3$. We expect that for some $c$ between 2 and 3, the curve $y = 2^x$ will pass through $(1,0)$ with slope exactly 1 (and this expectation can be justified using the Intermediate Value Theorem). Accordingly, we define $e$ to be the unique constant for which the curve $y = e^x$ has tangent line of slope exactly 1 at the point $(0,1)$. Thus by definition, if $f(x) = e^x$, then

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$$

and $e$ is the unique number with this property. (So we may take this as our definition of $e$.) It may be shown that $e \approx 2.71828\ldots$ .

If $f(x) = e^x$ then $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = \lim_{h \to 0} \frac{e^x e^h - e^x}{h}$$

$$= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$