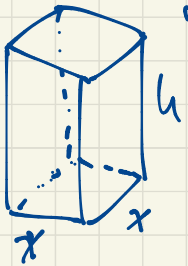
Math 2200-01 (Calculus I) Spring 2020

Book 3

Sec 4.5 : Optimization

April 13

p. 285 #19. of all boxes with a square base and a volume $8\,m^3$, which one has the minimum surface area?



volume $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}, \quad x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{4}{x^2}\left(x^3 - 8\right)$$

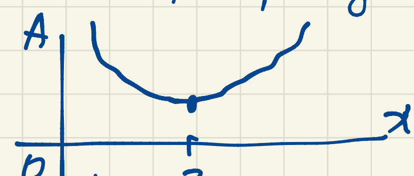
The critical point is at $x = 2$.

(Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing

For $x = 2$, $\frac{dA}{dx} = 0$.

For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.

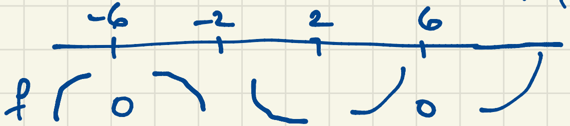


So the minimum surface area $A(2) = 12\,m^2$ occurs for a box of size $2m \times 2m \times 2m$.

Sec 4.4. p.278 #21. $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3+6x^2-36x-216$

$f'(x) = 3x^2+12x-36 = 3(x^2+4x-12) = 3(x+6)(x-2)$

$f''(x) = 6x+12 = 6(x+2)$

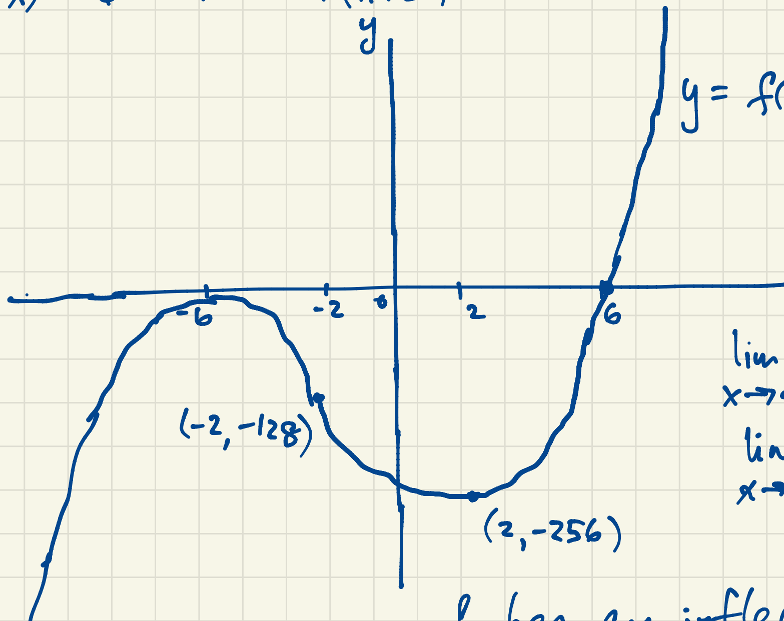


f' + + 0 − − − − 0 + + + +

f'' − − − − 0 + + + + + + +

$f(-2) = (-8)(4)^2 = -128$

$f(2) = (-4)(8^2) = -256$



$\lim_{x\to\infty} f(x) = \infty$

$\lim_{x\to-\infty} f(x) = -\infty$

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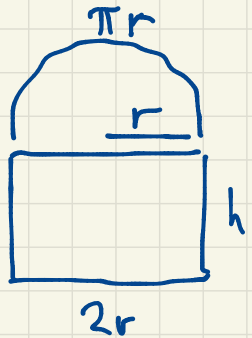
f is increasing on $(-\infty,-6)$ and on $(2,\infty)$,
decreasing on $(-6,2)$,
concave down on $(-\infty,-2)$,
concave up on $(-2,\infty)$.

f has an inflection point $(-2,-128)$,
a local minimum point $(2,-256)$,
a local maximum point $(-6,0)$,
no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.



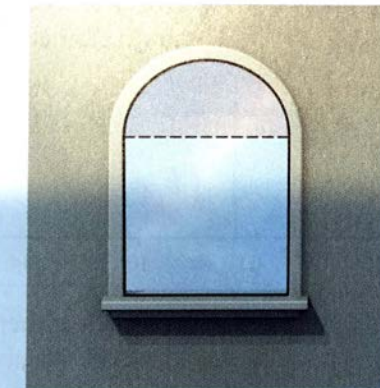
The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi+2)r + 2h = 20$$

$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi+2}{2} r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.



$$A = \frac{\pi}{2} r^2 + 2rh$$

$$= \frac{\pi}{2} r^2 + 2r\left(10 - \frac{\pi+2}{2} r\right)$$

$$= 20r + \left(\frac{\pi}{2} - (\pi+2)\right) r^2$$

$$= 20r - \frac{\pi+4}{2} r^2$$

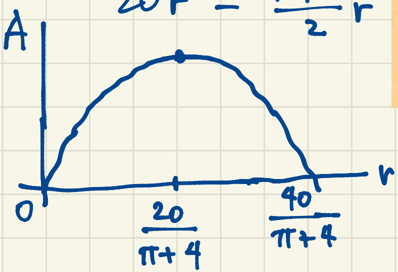
$$A = \left(20 - \frac{\pi+4}{2} r\right) r$$

$$\frac{dA}{dr} = 20 - (\pi+4) r$$

The critical point is at $r = \frac{20}{\pi+4}$.

When $0 < r < \frac{20}{\pi+4}$, $\frac{dA}{dr} > 0$ so A is increasing.

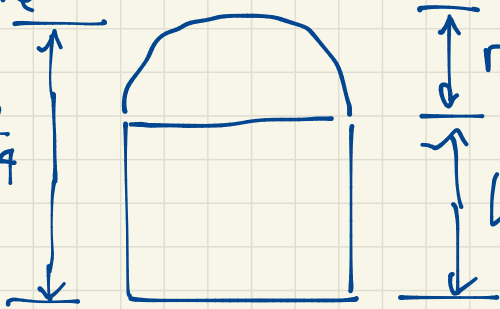
When $\frac{20}{\pi+4} < r < \frac{40}{\pi+4}$, $\frac{dA}{dr} < 0$ so A is decreasing.



So the maximum area occurs when $r = \frac{20}{\pi+4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi+4}\right]$, we need only check A at endpoints and the critical point.

The dimensions of the window that maximizes the area are

$\dfrac{40}{\pi+4}$



$r = \dfrac{20}{\pi+4}$

$2r = \dfrac{40}{\pi+4}$

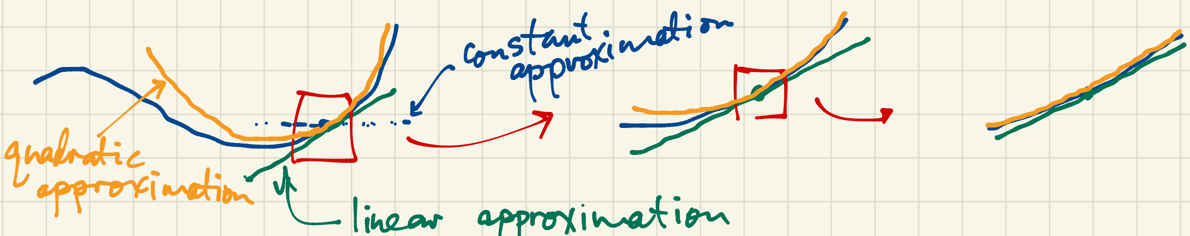
$$h = 10 - \frac{\pi+2}{2} r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$= \frac{20}{\pi+4}$$

Sec 4.6. Linearization and Differentials



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For x close to a, $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{}\,x + \boxed{}$

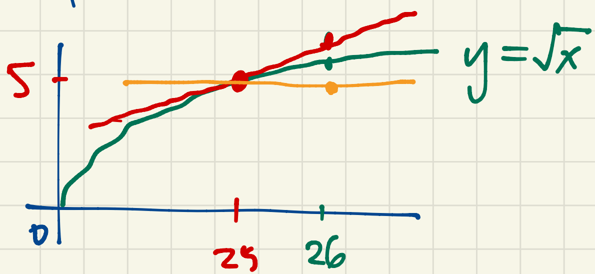
approximately equal to

linearization of f at $(a, f(a))$

$f'(a)$

$f(a) - af'(a)$

Example: Use the linearization of $\sqrt{x}$ at $(25, 5)$ to approximate $\sqrt{26}$.



$f(x) = \sqrt{x}$, $f(25) = 5$

$f'(x) = \frac{1}{2\sqrt{x}}$, $f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$

The linearization of $\sqrt{x}$ at $(25, 5)$ is

$$L(x) = f'(25)(x-25) + f(25)$$
$$= \frac{1}{10}(x-25) + 5.$$

If $x \approx 25$ then $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$.

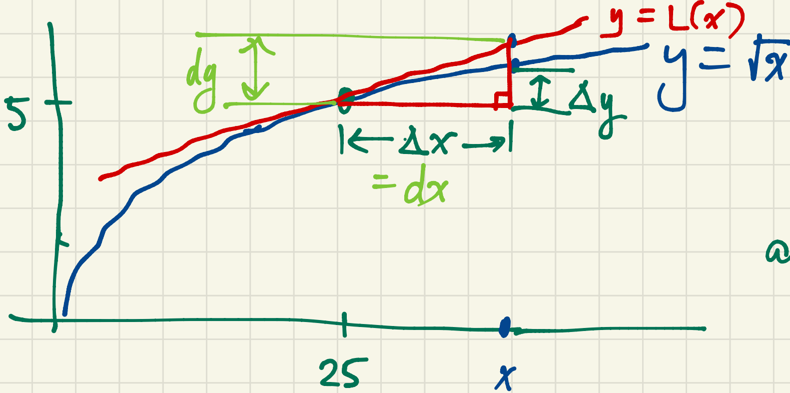
Eg. $\sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$

Check: $\sqrt{26} \approx 5.099019514$

correct to 3 significant digits.

Cruder approximation: $\sqrt{26} \approx 5$. (correct to one significant digit)

Constant approximation $\sqrt{x} \approx 5$



If we move from $(25, 5)$ to $(x, f(x))$ on the graph, our actual function $f(x) = \sqrt{x}$ changes by an exact amount

$$\Delta y = f(x) - f(25).$$

The secant line from $(25, 5)$ to $(x, f(x))$ has slope $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x) - f(25)}{x - 25}$.

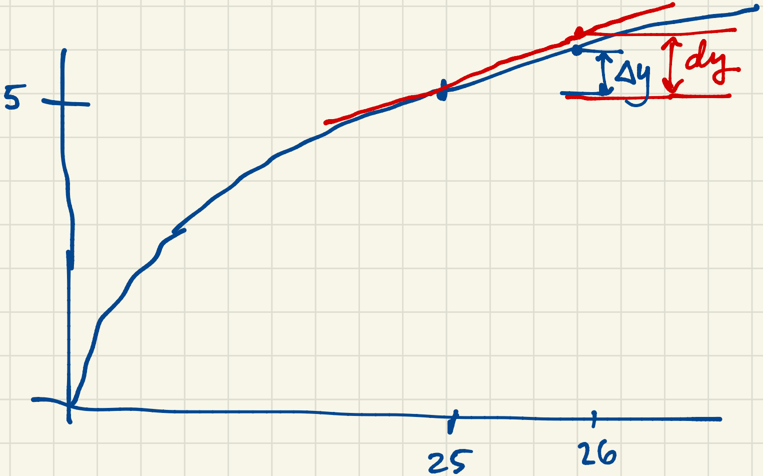
On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written $\frac{dy}{dx}$ as an indivisible symbol. Now we are interpreting dx and dy as changes in x and y. They are changes on the tangent line (just like Δx and Δy are corresponding changes on the actual curve of f).

dx and dy are <u>differentials</u>.

We'll interpret $\sqrt{26} \approx 5.1$ in this new language:



$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$

$dx = \Delta x = 26 - 25 = 1$

As we move from $x = 25$ to $x = 26$, the corresponding change in y is

$\Delta y \approx dy = 0.1$.

So $\sqrt{26} \approx 5.1$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals $\int_a^b f(x)\,\underbrace{dx}$

$x \longrightarrow u \longrightarrow y$

$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$

If $y = \sin x$, find $\frac{dy}{dx}$ and dy.

$\frac{dy}{dx} = \cos x$ so $dy = (\cos x)\, dx$

so $\Delta y \approx dy = \cos x \, dx$

Sec 4.7 l'Hôpital's Rule

The limit $\lim\limits_{x\to\infty} \underbrace{\dfrac{x^3+2x^2-x-1}{2x^3-5x}}_{\text{"indeterminate form } \frac{\infty}{\infty}\text{"}} = \lim\limits_{x\to\infty} \underbrace{\dfrac{1+\frac{2}{x}-\frac{1}{x^2}-\frac{1}{x^3}}{2-\frac{5}{x^2}}}_{\text{determinate form}} = \dfrac{1}{2}$.

The limit $\lim\limits_{x\to\infty} \dfrac{1}{3x^2+5} = 0$.

"determinate form $\frac{1}{\infty}$"

Do not confuse l'Hôpital's Rule with the Quotient Rule !

$$\lim_{x\to 0} \underbrace{\frac{\sin x}{x}}_{\text{indeterminate form } \frac{0}{0}} = 1$$

<u>l'Hôpital's Rule</u> For limits of indeterminate form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, $\lim\limits_{x} \dfrac{f(x)}{g(x)} = \lim\limits_{x} \dfrac{f'(x)}{g'(x)}$ assuming the second limit exists.

eg. $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = \lim\limits_{x\to 0} \dfrac{\cos x}{1} = 1$.

eg. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2-x-1}{2x^3-5x} = \lim\limits_{x\to\infty} \dfrac{3x^2+4x-1}{6x^2-5} = \lim\limits_{x\to\infty} \dfrac{6x+4}{12x} = \lim\limits_{x\to\infty} \dfrac{6}{12} = \dfrac{1}{2}$

eg. $\lim\limits_{x\to\infty} \underline{xe^{-x}} = \lim\limits_{x\to\infty} \frac{x}{e^x} = \lim\limits_{x\to\infty} \frac{1}{e^x} = 0$

indeterminate form $\infty\cdot 0$ indeterminate form $\frac{\infty}{\infty}$

eg. $\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} (-x) = 0.$

indeterminate form $0\cdot(-\infty)$ indeterminate $\frac{-\infty}{\infty}$

Don't keep using l'Hôpital's Rule beyond this point; that approach never reaches a conclusion.

$\lim\limits_{x\to 0^+} x\ln x = \lim\limits_{x\to 0^+} \frac{x}{\frac{1}{\ln x}} = \lim\limits_{x\to 0^+} \frac{1}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = \lim\limits_{x\to 0^+} \left(-x(\ln x)^2\right)$ this is not an improvement!

indeterminate form $\frac{0}{0}$

$(a-b)(a+b) = a^2 - b^2$

$\lim\limits_{x\to\infty} \left(\sqrt{x^2+10x} - x\right)\cdot\frac{\sqrt{x^2+10x}+x}{\sqrt{x^2+10x}+x} = \lim\limits_{x\to\infty} \frac{x^2+10x-x^2}{\sqrt{x^2+10x}+x} = \lim\limits_{x\to\infty} \frac{10x}{\sqrt{x^2+10x}+x}$

indeterminate form $\infty - \infty$

$= \lim\limits_{x\to\infty} \frac{10}{\sqrt{1+\frac{10}{x}}+1} = \frac{10}{\sqrt{1+0}+1} = 5.$

Note:
$\sqrt{x^2+10x} = \sqrt{x^2(1+\frac{10}{x})}$
$= x\sqrt{1+\frac{10}{x}}$
$\sqrt{x^2} = |x|$
$= \begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$

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Another l'Hôpital's Rule example for the indeterminate form 1^∞.

$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x$ where a is constant.

In order to use l'Hôpital's Rule, we need to rewrite this as a quotient.

$b^c = (e^{\ln b})^c = e^{c\ln b}$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{a}{x}\right)} = \lim_{u\to 1} e^{\frac{a}{u-1}\ln u} = e^{a\cdot 1} = e^a.$$

$u = 1+\frac{a}{x} \to 1.$

$$\lim_{u\to 1}\frac{\ln u}{u-1} = \lim_{u\to 1}\frac{1/u}{1} = 1.$$

$u-1 = \frac{a}{x}$

(indeterminate form $\frac{0}{0}$)

$x = \frac{a}{u-1}$

If $f(x) = e^{ax}$ then $\lim_{u\to 1} f\left(\frac{\ln u}{u-1}\right) = f\left(\lim_{u\to 1}\frac{\ln u}{u-1}\right) = f(1) = e^{a\cdot 1} = e^a.$

Does $\lim_{u} f(g(u)) = f\left(\lim_{u} g(u)\right)$? Can you move the limit inside?

We can do this when f is continuous and g is continuous.

$c = \lim_{u} g(u)$, write $t = g(u) \to c$. $\lim_{t\to c} f(t) = f(c)$.

Some books define $e^a = \lim_{x\to\infty} \left(1+\frac{a}{x}\right)^x$.

This limit comes from compound interest.

If you deposit a principal amount A in the bank at nominal interest rate r (eg. 5% interest per annum would give $r=0.05$). If interest is compounded annually then after one year you have earned rA interest. The ~~total~~ balance in the bank after a year would be $A + rA = (1+r)A$.

If interest is compounded semiannually (every 6 months) then after 6 months you have $(1+\frac{r}{2})A$ as balance after 6 months; then at the end of the year you have

$$\left(1+\frac{r}{2}\right)\cdot\left(1+\frac{r}{2}\right)A = \left(1+\frac{r}{2}\right)^2 A = \Big(1+\underbrace{r+\frac{r^2}{4}}_{\text{effective rate of interest.}}\Big)A.$$

If interest is compounded n times per year then every $\frac{1}{n}$ year your balance is multiplied by $1+\frac{r}{n}$. After one year your balance is $\left(1+\frac{r}{n}\right)^n A$.

For continually compounded interest we let $n\to\infty$.

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n A = e^r A.$$

Eg. 5% interest compounded continuously results in balance $e^{0.05}A \approx 1.05127\,A$ (effectively 5.127 % interest rate per annum).

p. 311 # 69. $\displaystyle \lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \lim_{x\to\infty} \frac{\cancel{(\ln x)}/\ln 2}{\cancel{(\ln x)}/\ln 3} = \frac{\ln 3}{\ln 2} = \log_2 3$

$\log_2 x = \dfrac{\ln x}{\ln 2}$ why?

$$a = \log_2 x \iff 2^a = x \iff \ln(2^a) = \ln x$$

$$\iff a \ln 2 = \ln x$$

$$\iff a = \frac{\ln x}{\ln 2}.$$

$$\begin{tikzcd} x \arrow[r, bend left, "\log_2"] & a \arrow[l, bend left, "\exp_2"] \end{tikzcd}$$