

Math 2200-01 (Calculus I) Spring 2020

Book 3



Sec 4.5: Optimization

April 13

p.285 #19. of all boxes with a square base and a volume 8m^3 , which one has the minimum surface area?



volume $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

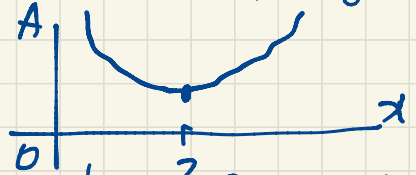
area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}, x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{1}{x^2} (x^3 - 8)$$

The critical point is at $x=2$.
 (Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing.
 For $x=2$, $\frac{dA}{dx} = 0$.
 For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.

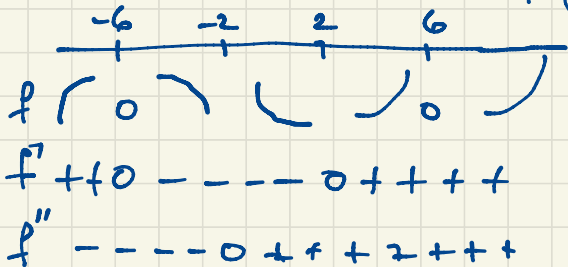


So the minimum surface area $A(2) = 12\text{m}^2$ occurs for a box of size $2\text{m} \times 2\text{m} \times 2\text{m}$.

Sec 4.4. p. 278 # 21. $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$

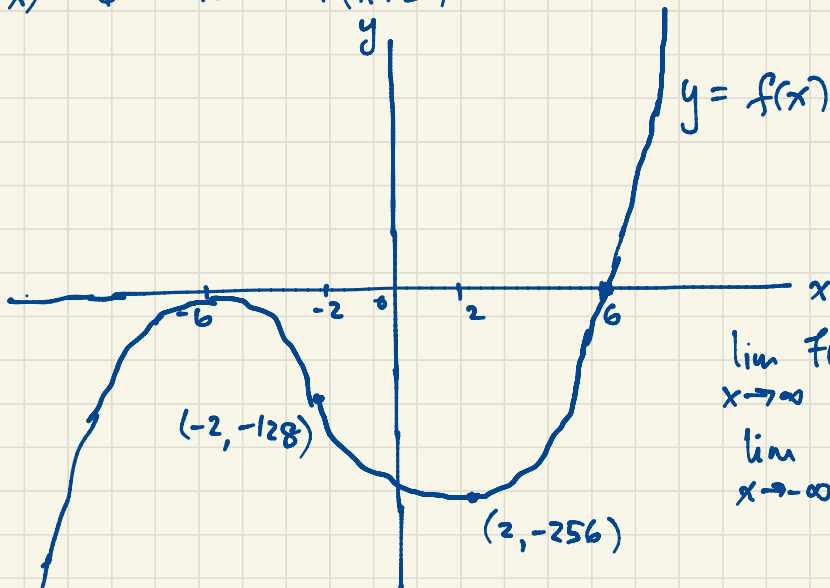
$$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$$

$$f''(x) = 6x + 12 = 6(x+2)$$



$$f(-2) = (-8)(4)^2 = -128$$

$$f(2) = (-4)(8^2) = -256$$

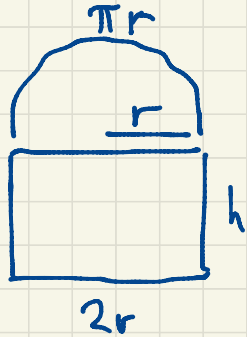


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f is increasing on $(-\infty, -6)$ and on $(2, \infty)$,
 decreasing on $(-6, 2)$,
 concave down on $(-\infty, -2)$,
 concave up on $(-2, \infty)$.

f has an inflection point $(-2, -128)$,
 a local minimum point $(2, -256)$,
 a local maximum point $(-6, 0)$,
 no absolute extrema or asymptotes.

Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.



The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi + 2)r + 2h = 20$$

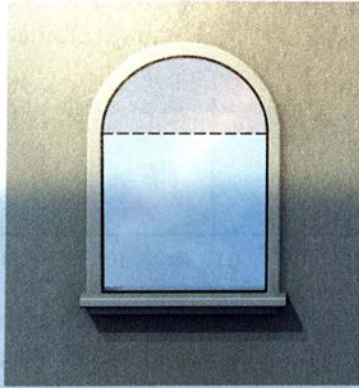
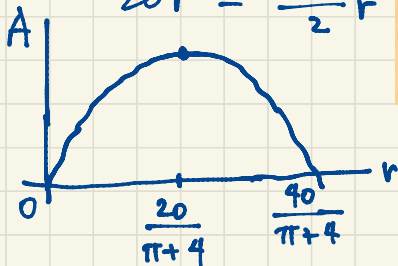
$$2h = 20 - \pi r - 2r$$

$$h = 10 - \frac{\pi + 2}{2}r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.

$$\frac{dA}{dr} = 20 - (\pi + 4)r$$

$$\begin{aligned} A &= \frac{\pi}{2}r^2 + 2rh \\ &= \frac{\pi}{2}r^2 + 2r\left(10 - \frac{\pi + 2}{2}r\right) \\ &= 20r + \left(\frac{\pi}{2} - (\pi + 2)\right)r^2 \\ &= 20r - \frac{\pi + 4}{2}r^2 \end{aligned}$$



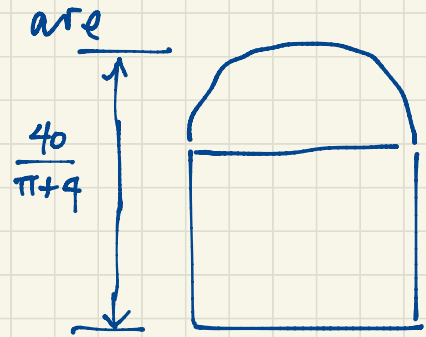
The critical point is at $r = \frac{20}{\pi + 4}$.
When $0 < r < \frac{20}{\pi + 4}$, $\frac{dA}{dr} > 0$ so A is increasing.

When $\frac{20}{\pi + 4} < r < \frac{40}{\pi + 4}$, $\frac{dA}{dr} < 0$ so A is decreasing.

$$A = \left(20 - \frac{\pi + 4}{2}r\right)r$$

So the maximum area occurs when $r = \frac{20}{\pi + 4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi + 4}\right]$, we need only check A at endpoints and the critical point.

The dimensions of the window that maximizes the area



$$r = \frac{20}{\pi+4}$$

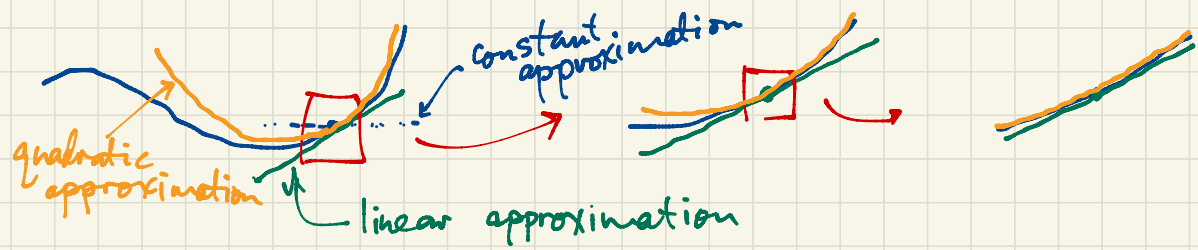
$$h = 10 - \frac{\pi+2}{2}r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$

$$= 10 - \frac{10(\pi+2)}{\pi+4}$$

$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$

$$2r = \frac{40}{\pi+4}$$

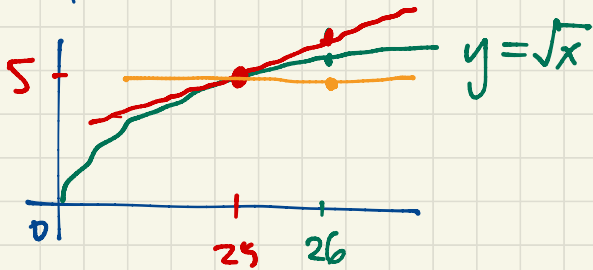
Sec 4.6. Linearization and Differentials $= \frac{20}{\pi+4}$



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For x close to a , $f(x) \approx L(x) = f'(a)(x-a) + f(a) = \boxed{}x + \boxed{}$
 approximately equal to $\underbrace{}_{\text{linearization of } f \text{ at } (a, f(a))}$
 $f'(a)$ $f(a) - af'(a)$

Example: Use the linearization of \sqrt{x} at $(25, 5)$ to approximate $\sqrt{26}$.



$$f(x) = \sqrt{x}, \quad f(25) = 5$$

$$f'(x) = \frac{1}{2\sqrt{x}}, \quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

The linearization of \sqrt{x} at $(25, 5)$ is

$$\begin{aligned} L(x) &= f'(25)(x-25) + f(25) \\ &= \frac{1}{10}(x-25) + 5. \end{aligned}$$

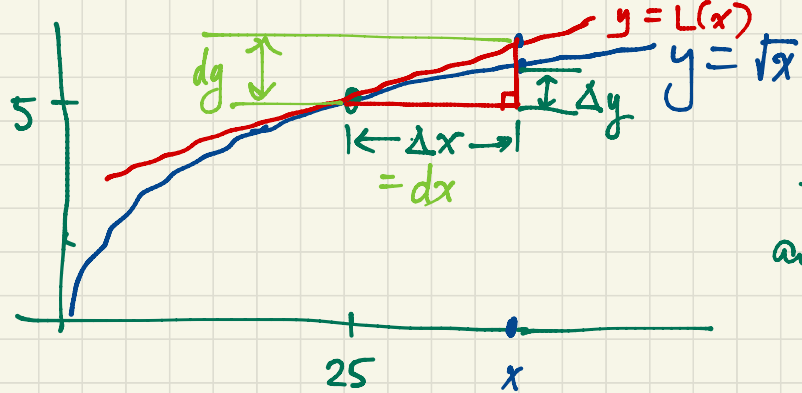
If $x \approx 25$ then $\sqrt{x} \approx \frac{1}{10}(x-25) + 5$.

$$\text{Eq. } \sqrt{26} \approx \frac{1}{10}(26-25) + 5 = 5.1$$

$$\text{Check: } \sqrt{26} \approx 5.099019514$$

correct to 3 significant digits.

Coarser approximation: $\sqrt{26} \approx 5$. (correct to one significant digit)
 Constant approximation $\sqrt{x} \approx 5$



If we move from $(25, 5)$ to $(x, f(x))$ on the graph, our actual function $f(x) = \sqrt{x}$ changes by an exact amount

$$\Delta g = f(x) - f(25).$$

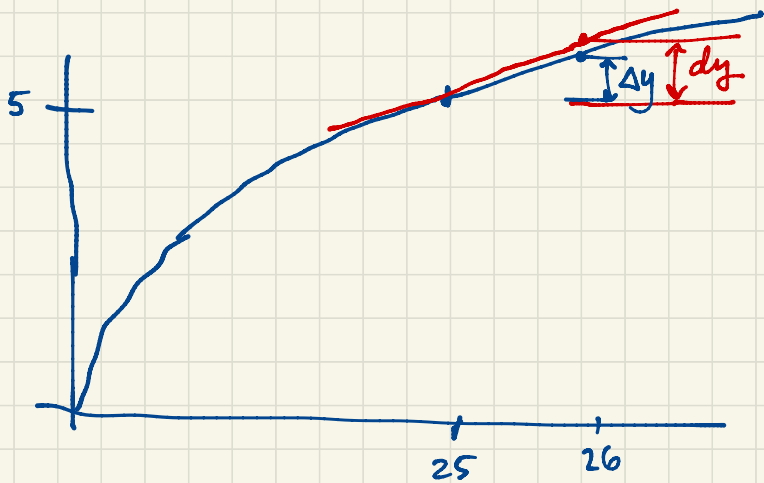
The secant line from $(25, 5)$ to $(x, f(x))$ has slope $\frac{\Delta g}{\Delta x} = \frac{f(x) - f(25)}{x - 25}$.

On the tangent line,

$$\frac{\Delta L(x)}{\Delta x} = \frac{L(x) - L(25)}{x - 25} = f'(25) = \frac{dy}{dx}$$

Until now we have written $\frac{dy}{dx}$ as an indivisible symbol. Now we are interpreting dx and dy as changes in x and y . They are changes on the tangent line (just like Δx and Δy are corresponding changes on the actual curve of f). dx and dy are differentials.

We'll interpret $\sqrt{26} \approx 5.1$ in this new language:



$$f'(25) = \frac{1}{10} = \frac{dy}{dx} \Rightarrow dy = \frac{1}{10} dx = \frac{1}{10} \times 1 = 0.1$$

$$dx = \Delta x = 26 - 25 = 1$$

As we move from $x=25$ to $x=26$,
the corresponding change in y is

$$\Delta y \approx dy = 0.1.$$

$$\text{So } \sqrt{26} \approx 5.1$$

This is a quick and easy interpretation for differentials. We will be using differentials throughout Calculus.

Integrals $\int_a^b f(x) dx$

$$x \rightarrow u \rightarrow y$$

$$\frac{dy}{dx} = \frac{dy}{\cancel{du}} \cdot \frac{\cancel{du}}{dx}$$

If $y = \sin x$, find $\frac{dy}{dx}$ and dy .

$$\frac{dy}{dx} = \cos x \quad \text{so} \quad dy = (\cos x) dx$$

$$\text{so } \Delta y \approx dy = \cos x dx$$