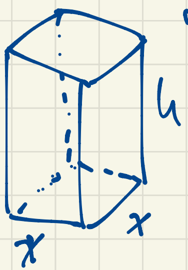
Math 2200-01 (Calculus I) Spring 2020

Book 3

Sec 4.5 : Optimization

April 13

p.285 #19. Of all boxes with a square base and a volume $8\,m^3$, which one has the minimum surface area?



volume $V = x^2 h = 8 \Rightarrow h = \frac{8}{x^2}$

area $A = \underbrace{2x^2}_{\text{top and bottom}} + \underbrace{4xh}_{\text{sides}} = 2x^2 + 4x \cdot \frac{8}{x^2} = 2x^2 + \frac{32}{x}$, $x > 0$

The domain is $(0, \infty)$, an unbounded open interval.

$$\frac{dA}{dx} = 4x - \frac{32}{x^2} = \frac{4}{x^2}\left(x^3 - 8\right)$$

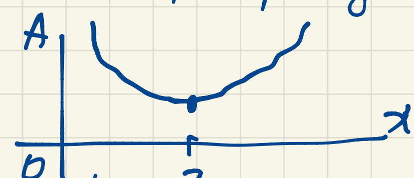
The critical point is at $x = 2$.

(Recall: critical points are where the derivative is zero or undefined. There are no points of the domain where $\frac{dA}{dx}$ is undefined, only one point where $\frac{dA}{dx} = 0$.)

For $0 < x < 2$, $\frac{dA}{dx} < 0$ so $A(x)$ is decreasing.

For $x = 2$, $\frac{dA}{dx} = 0$.

For $x > 2$, $\frac{dA}{dx} > 0$ so $A(x)$ is increasing.



So the minimum surface area $A(2) = 12\,m^2$ occurs for a box of size $2m \times 2m \times 2m$.

Sec 4.4. p.278 # 21. $f(x) = (x-6)(x+6)^2 = (x^2-36)(x+6) = x^3 + 6x^2 - 36x - 216$

$f'(x) = 3x^2 + 12x - 36 = 3(x^2 + 4x - 12) = 3(x+6)(x-2)$

$f''(x) = 6x + 12 = 6(x+2)$

$-6 \qquad -2 \qquad 2 \qquad 6$

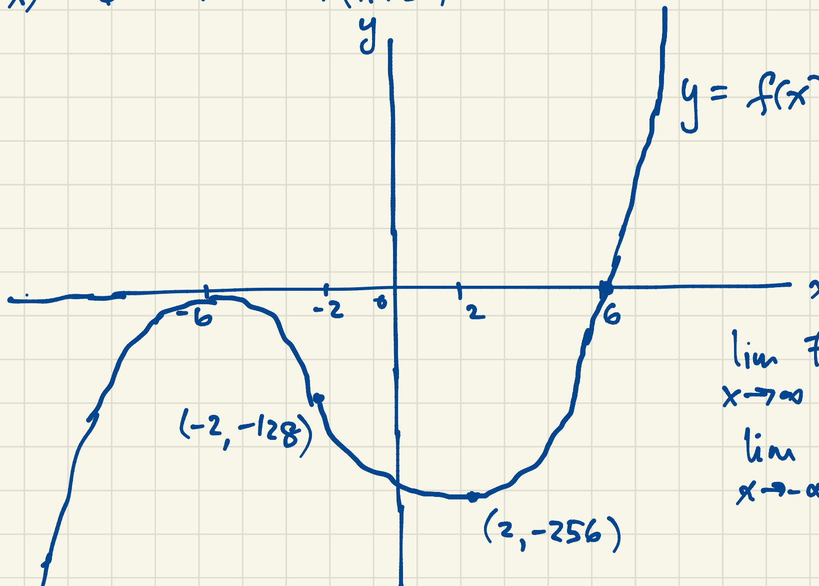
f 0   0

f' + + 0 − − − − − 0 + + + +

f'' − − − − 0 + + + + + + +

$f(-2) = (-8)(4)^2 = -128$

$f(2) = (-4)(8^2) = -256$



$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

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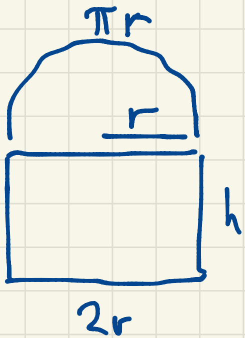
f is increasing on $(-\infty, -6)$ and on $(2, \infty)$,
decreasing on $(-6, 2)$,
concave down on $(-\infty, -2)$,
concave up on $(-2, \infty)$.

f has an inflection point $(-2, -128)$,
a local minimum point $(2, -256)$,
a local maximum point $(-6, 0)$,
no absolute extrema or asymptotes.

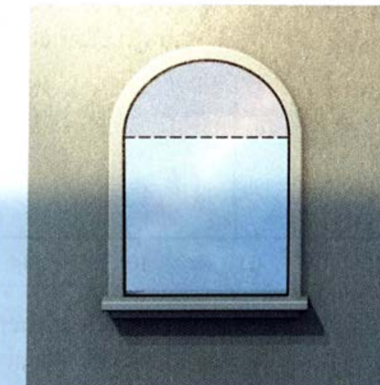
Sec 4.5 p. 287 #41. Let r be the radius of the semicircular window pane.



The perimeter is $P = \pi r + 2r + 2h = 20$

$$(\pi+2)r + 2h = 20$$
$$2h = 20 - \pi r - 2r$$
$$h = 10 - \frac{\pi+2}{2} r$$

41. A window consists of rectangular pane of glass surmounted by a semicircular pane of glass (see figure). If the perimeter of the window is 20 feet, determine the dimensions of the window that maximize the area of the window.



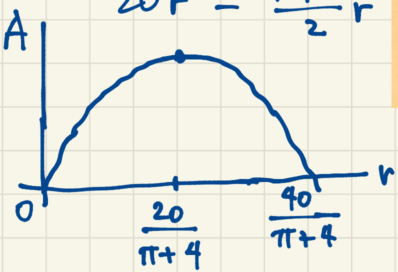
$$\frac{dA}{dr} = 20 - (\pi+4)r$$

$$A = \frac{\pi}{2} r^2 + 2rh$$
$$= \frac{\pi}{2} r^2 + 2r\left(10 - \frac{\pi+2}{2} r\right)$$
$$= 20r + \left(\frac{\pi}{2} - (\pi+2)\right) r^2$$
$$= 20r - \frac{\pi+4}{2} r^2$$

The critical point is at $r = \frac{20}{\pi+4}$.

When $0 < r < \frac{20}{\pi+4}$, $\frac{dA}{dr} > 0$ so A is increasing.

When $\frac{20}{\pi+4} < r < \frac{40}{\pi+4}$, $\frac{dA}{dr} < 0$ so A is decreasing.

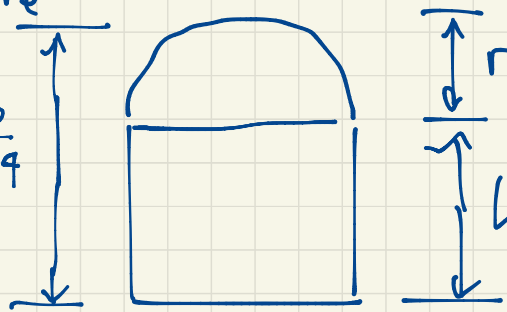


$$A = \left(20 - \frac{\pi+4}{2} r\right) r$$

So the maximum area occurs when $r = \frac{20}{\pi+4}$. Alternatively since $A \geq 0$ requires r to be in $\left[0, \frac{40}{\pi+4}\right]$, we need only check A at endpoints and the critical point.

The dimensions of the window that maximizes the area are

$\frac{40}{\pi+4}$



$$r = \frac{20}{\pi+4}$$

$$h = 10 - \frac{\pi+2}{2} r = 10 - \frac{\pi+2}{2} \cdot \frac{20}{\pi+4}$$
$$= 10 - \frac{10(\pi+2)}{\pi+4}$$
$$= \frac{10(\pi+4) - 10(\pi+2)}{\pi+4}$$
$$= \frac{20}{\pi+4}$$

$$2r = \frac{40}{\pi+4}$$

Sec 4.6. Linearization and Differentials