Name ....

Section ☐ **01** Earl-8am ☐ **02** Yeung-9am ☐ **03** Furtado-10am ☐ **04** Li-11am
☐ **05** Furtado-11am ☐ **06** Zhong-12noon ☐ **07** Wiseman-1:10pm ☐ **08** Yeung-2:10pm

UNIVERSITY OF WYOMING

Math 2200 — Spring 2019

Calculus I

Department of Mathematics

Common Exam III

5:15–7:00pm Thursday April 25, 2019

Instructions. Indicate your name and section/instructor above. You may use a scientific non-graphing calculator—no other aids are allowed. Cell phones and other devices must be turned off and left in your backpack/bag during the exam. **Write clearly**, using good mathematical notation and showing all required steps in the space provided. *Unless otherwise stated, justify your answers.* A list of useful formulas appears on the last page. Total value: 100 points.

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1. *(15 points)* Air is pumped into a spherical balloon at a rate of 20 cubic centimeters per second. How fast is the radius of the balloon increasing, at the instant when its radius is 50cm? (Recall that a sphere of radius r has volume $V = \frac{4}{3}\pi r^3$.)

Given: $\frac{dV}{dt} = 20$.

Find: $\left.\frac{dr}{dt}\right|_{r=50} = ?$

Taking derivative with respect to the time on $V = \frac{4}{3}\pi r^3$, we have

$$\frac{dV}{dt} = \frac{4}{3}\pi \frac{dr^3}{dt}.$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt}$$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}.$$

So $\left.\frac{dr}{dt}\right|_{r=50} = \frac{20}{4\pi(50^2)}$

$= \frac{1}{500\pi}$ (cm/s).

2. *(18 points)* Let $f(x) = x(x-3)^2$.

(a) Find the values of x where $f'(x) = 0$.

$$f'(x) = [x(x-3)^2]' = [x]'(x-3)^2 + x[(x-3)^2]'$$
$$= (x-3)^2 + 2x(x-3) = (x-3)[(x-3)+2x].$$
$$= (x-3)[3x-3] = 3(x-3)(x-1).$$

So $f'(x) = 0$ at $x = 1$ and $x = 3$.

(b) On which interval(s) is $f(x)$ increasing?

$(-\infty, 1)$: $f'(x) > 0$ so $f(x) \uparrow$

$(1, 3)$: $f'(x) < 0$ so $f(x) \downarrow$.

$(3, \infty)$: $f'(x) > 0$ so $f(x) \uparrow$.

Conclusion: $f(x)$ is increasing on $(-\infty, 1)$ and $(3, \infty)$.

(c) On which interval(s) is $f(x)$ decreasing?

$f(x)$ is decreasing on $(1, 3)$.

(d) Find the values of x where $f''(x) = 0$.

$$f''(x) = [3(x-3)(x-1)]'$$
$$= 3[(x-3)(x-1)]' = 3[(x-3)'(x-1) + (x-3)(x-1)']$$
$$= 3[x-1 + x-3] = 3[2x-4]$$
$$= 6(x-2).$$

So $f''(x) = 0$ at $x = 2$.

(e) On which interval(s) is the graph of f concave up?

$(-\infty, 2)$: $f''(x) < 0$ so $f(x)$ is CD

$(2, \infty)$: $f''(x) > 0$ so $f(x)$ is CU.

Conclusion. $f(x)$ is concave up on $(2, \infty)$.

(f) On which interval(s) is the graph of f concave down?

$f(x)$ is concave down on $(-\infty, 2)$.

3. *(12 points)* In each case, evaluate the indicated limit analytically. (You may use the limit laws or l'Hôpital's Rule; but *calculator-based estimates will not be accepted*.)

(a) $\lim_{x\to 0^+} \dfrac{\cos(x)}{\cos(x^2)}$

$$= \frac{\cos(0)}{\cos(0^2)} = \frac{1}{1} = 1.$$

(b) $\lim_{x\to 1} \dfrac{(x-1)e^x}{\ln(x)}$

Since $e^x \to e^1$ as $x \to 1$, we can simplify the limit as

$$\lim_{x\to 1} \frac{(x-1)e^x}{\ln x} = \lim_{x\to 1} \frac{(x-1)e}{\ln x}$$

L'Hopital's rule

$$= e \lim_{x\to 1} \frac{x-1}{\ln x} = e \lim_{x\to 1} \frac{(x-1)'}{(\ln x)'}$$

$$= e \lim_{x\to 1} \frac{1}{\frac{1}{x}} = e \lim_{x\to 1} x = e.$$

(c) $\lim_{x\to\infty} \dfrac{\sqrt{3x^2+10}}{x-30}$

$$= \lim_{x\to\infty} \frac{\sqrt{x^2\left[3+\frac{10}{x^2}\right]}}{x\left(1-\frac{30}{x}\right)}.$$

$$= \lim_{x\to\infty} \frac{x\sqrt{3+\frac{10}{x^2}}}{x\left(1-\frac{30}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{3+\frac{10}{x^2}}}{1-\frac{30}{x}} = \frac{\sqrt{3+0}}{1-0}$$

$$= \sqrt{3}.$$

4. *(8 points)* A car, travelling along a straight road, has position $s(t)$ (in miles) at time t (in hours) for $0 \leqslant t \leqslant 1$. Assume that $s(0) = 0$ and $s(1) = 60$, so that the car travels 60 miles in one hour. *In each of the following, circle A, B, C or D to indicate the best choice available.* Refer to the Mean Value Theorem (MVT) for Derivatives.

(i) We may presume that the function $s(t)$ is continuous

A. based on physical considerations.

B. by the MVT.

C. since the car is moving steadily forward.

D. since $s(t)$ is necessarily a polynomial in t.

(ii) The average velocity of the car during the one-hour time interval is

A. unknown.

B. 60 miles per hour.

C. $s'(t)$.

D. none of the above, but derivable using the MVT.

(iii) The main assertion of the MVT is that

A. at some time t between 0 and 1, the car was moving faster than at any other time during the hour.

B. the car reached its maximum velocity at one, and only one, instant during the hour.

C. the car reached its average velocity at one, and only one, instant during the hour.

D. at some time between 0 and 1, the instantaneous velocity of the car was exactly 60 miles per hour.

(iv) The statement of the MVT may be viewed as

A. a formula for computing the instantaneous velocity of the car.

B. a formula indicating when the instantaneous velocity of the car reaches its maximum or minimum.

C. a guarantee that at some instant, the velocity agrees with the average velocity during the hour.

D. a formula indicating when the instantaneous velocity of the car reaches its average value.

5. *(10 points)* A differentiable function f, satisfying $f(1) = 3$ and $f'(1) = 2$, is approximated by its linearization $f(x) \approx L(x)$ near the point $(1, 3)$.

(a) Explicitly determine the linear function $L(x)$.

$$L(x) = f(1) + f'(1)(x-1)$$

$$= 3 + 2(x-1)$$

(b) Based on the linear approximation, *estimate* the value of $f(3)$.

The linear approximation of $f(x)$ at $x=1$ is

$$f(x) \approx L(x) = 3 + 2(x-1).$$

So

$$f(3) \approx 3 + 2(3-1) = 7$$

(c) Is it possible for the estimate in (b) to be exact? Answer 'yes' or 'no' with a brief explanation.

Yes, it is possible. For example, when the curve $f(x)$ is a straight line, $f(x) = L(x)$.

(d) Is it possible for the estimate in (b) to be off by more than 10? Answer 'yes' or 'no' with a brief explanation.

Yes, it is possible. $L(x)$ is the tangent line to the curve $f(x)$ at $x=1$. When x is far away from $x=1$, say, $x=3$, $f(x)$ can be far away from $L(x)$.

6. *(15 points)* Tests show that if a car is driven at a constant speed of x miles per hour, it can be expected to travel a distance of

$$f(x) = \frac{Ax}{c^2 + x^2}$$

miles before its engine fails; here A and c are positive constants.

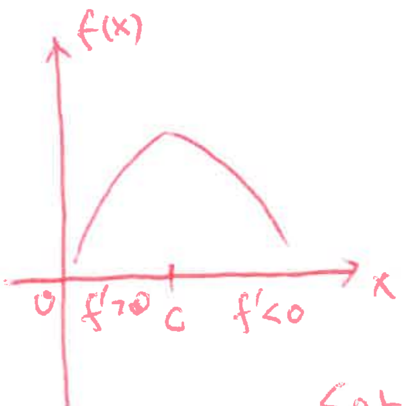
(a) At what speed should the car be driven, in order to maximize the distance the car can travel during its lifetime? Show your work, explaining why your solution *maximizes* the distance travelled.

Find critical point(s):

$$f'(x) = \left[\frac{Ax}{c^2+x^2}\right]' = \frac{[Ax]'(c^2+x^2) - Ax(c^2+x^2)'}{(c^2+x^2)^2}$$

$$= \frac{A(c^2+x^2) - Ax(2x)}{(c^2+x^2)^2}$$

$$= \frac{Ac^2 + Ax^2 - 2Ax^2}{(c^2+x^2)^2} = \frac{Ac^2 - Ax^2}{(c^2+x^2)^2}.$$



Set $f'(x) = 0$ and solve

$$\frac{Ac^2 - Ax^2}{(c^2+x^2)^2} = 0 \Rightarrow Ac^2 - Ax^2 = 0 \Rightarrow x = c.$$

Since $f'(x)$ changes sign from positive to negative at $x=c$, $f(x)$ has a maximum at $x = c$

(b) What is the maximum distance the car can be expected to travel during its lifetime?

The maximum distance is

$$f(c) = \frac{Ac}{c^2+c^2} = \frac{Ac}{2c^2}.$$

$$= \frac{A}{2c}.$$

7. *(12 points)* Determine the absolute maximum and minimum values of the function

$$h(x) = x^4 - 18x^2 + 1, \qquad 0 \leqslant x \leqslant 4.$$

Also indicate where these values occur. Show your work, justifying why your answers give the absolute extreme values of h.

Absolute maximum: $h(\ 0\ ) = 1$

Absolute minimum: $h(\ 3\ ) = -80$

Use the closed interval method to find extreme value of $h(x)$ on $[0,4]$:

Step #1. Find interior critical points

$$h'(x) = 4x^3 - 36x = 0$$

$$\Rightarrow \quad 4x(x^2-9) = 0$$

$$\Rightarrow \quad 4x(x+3)(x-3) = 0$$

$$\Rightarrow \quad x=0, \quad x=3, \quad x=-3 \text{ (rejected)}.$$

Step #2. Evaluate function values.

$$h(0) = 0^4 - 18(0^2) + 1 = 1$$

$$h(3) = 3^4 - 18(3^2) + 1 = -80$$

$$h(4) = 4^4 - 18(4^2) + 1 = -31.$$

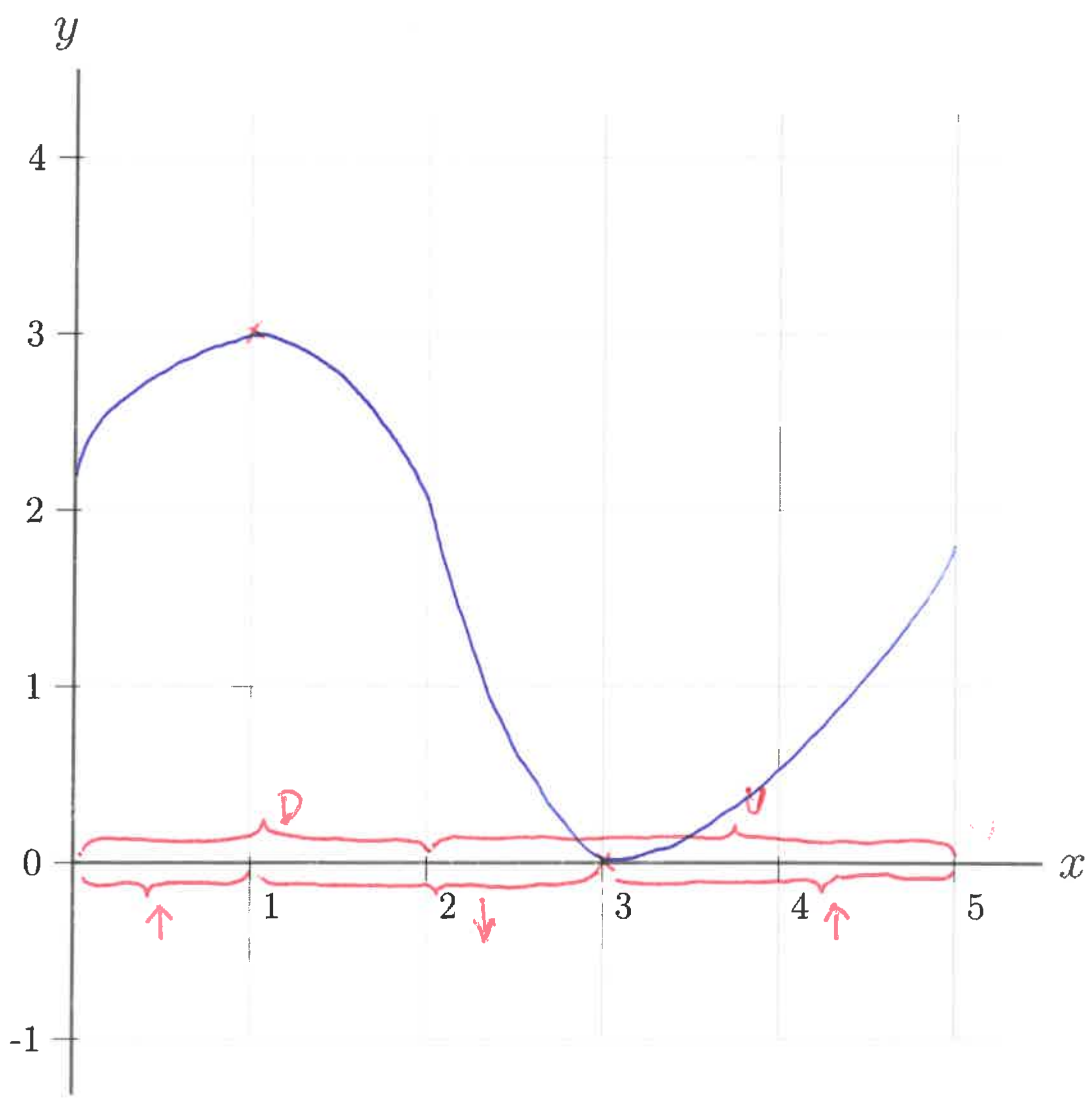
Step #3. Conclusion.

Absolute maximum is $h(0) = 1$.

Absolute minimum is $h(3) = -80$.

8. *(10 points)* Using the grid below, sketch the graph of a function $h(x)$ that is continuous on the interval $[0, 5]$ and that has all six of the following properties:

$h(1) = 3.$
$h(3) = 0.$
$h' > 0$ on the intervals $(0, 1)$ and $(3, 5)$.
$h' < 0$ on the interval $(1, 3)$.
$h'' > 0$ on the interval $(2, 5)$.
$h'' < 0$ on the interval $(0, 2)$.



(SCRATCH WORK)

Useful Formulas

$f(u)$	$f'(u)$
$\tan u$	$\sec^2 u$
$\cot u$	$-\csc^2 u$
$\sec u$	$\sec u \tan u$
$\csc u$	$-\csc u \cot u$
$\ln f(u)$	$f'(u)/f(u)$
$\log_b u$	$1/(u \ln b)$
$\sin^{-1} u$	$1/\sqrt{1-u^2}$
$\tan^{-1} u$	$1/(1+u^2)$
$\sec^{-1} u$	$1/(\lvert u\rvert\sqrt{u^2-1})$
$\cos^{-1} u$	$-1/\sqrt{1-u^2}$
$\cot^{-1} u$	$-1/(1+u^2)$
$\csc^{-1} u$	$-1/(\lvert u\rvert\sqrt{u^2-1})$

Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$

For instructors' use only:

Question	1	2	3	4	5	6	7	8	Total
Points	15	18	12	8	10	15	12	10	100
Score									