Name Solution Key

Section ☐ **01** Earl-8am ☐ **02** Yeung-9am ☐ **03** Furtado-10am ☐ **04** Li-11am ☐ **05** Furtado-11am ☐ **06** Zhong-12noon ☐ **07** Wiseman-1:10pm ☐ **08** Yeung-2:10pm

UNIVERSITY OF WYOMING

$\frac{d}{dx} \sin x = \cos x$

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Math 2200 — Spring 2019

Calculus I

Department of Mathematics

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

(Final) Common Exam IV

1:15–3:15pm Wednesday May 15, 2019

Instructions. Indicate your name and section/instructor above. You may use a scientific non-graphing calculator—no other aids are allowed. Cell phones and other devices must be turned off and left in your backpack/bag during the exam. **Write clearly**, using good mathematical notation and showing all required steps in the space provided. *Unless otherwise stated, justify your answers.* A list of useful formulas appears on the last page. Total value: 100 points.

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1. *(8 points)* At time $t \geqslant 0$ (in minutes), a coffee mug has temperature given by

$$T(t) = 70(1 + e^{-t/10})$$

degrees Fahrenheit.

(a) At what rate is the temperature cooling at time $t = 0$? Answer using appropriate units.

$T'(t) = 70(-\frac{1}{10}e^{-t/10}) = -7e^{-t/10}$

$T'(0) = -7$ degrees per minute

(b) What eventually happens to the temperature of the coffee as time progresses? Answer using an appropriate limit.

Since $\lim_{t\to\infty} T(t) = 70$, the temperature of the coffee approaches 70 degrees Fahrenheit as time goes on.

2. *(8 points)* Let $f(x) = e^{3x} + x^2$.

(a) Find *all* differentiable functions F satisfying $F' = f$. (There are infinitely many solutions.)

$F(x) = \int (e^{3x} + x^2)\,dx = \frac{1}{3}e^{3x} + \frac{1}{3}x^3 + C$ where C is an arbitrary constant.

(b) Among all functions F satisfying $F' = f$, determine the solution which also satisfies $F(0) = 0$.

For an antiderivative $F(x)$ as above, satisfying $F(0) = 0 = \frac{1}{3} + C$, we have $C = -\frac{1}{3}$, so $F(x) = \frac{1}{3}e^{3x} + \frac{1}{3}x^3 - \frac{1}{3}$

3. *(8 points)* The limit

$$\lim_{x \to \pi} \frac{e^{\sin(x)} - 1}{x - \pi}$$

expresses the derivative $f'(a)$ for some function f and constant a.

(a) Determine a function $f(x)$ and a constant a for which the derivative takes the indicated form *by definition*.

We may take $f(x) = e^{\sin x}$ and $a = \pi$; then

$$f'(\pi) = \lim_{x \to \pi} \frac{f(x) - f(\pi)}{x - \pi} = \lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi}.$$

(b) Evaluate the limit by any analytic method you know. (In particular, you may simply evaluate the derivative found in (a), or use l'Hôpital's Rule. Do not rely on guesses based on calculator estimates).

Since $f'(x) = e^{\sin x} \cos x$, we have $f'(\pi) = 1 \cdot (-1) = -1$.

Alternatively by l'Hôpital's Rule,

$$\lim_{x \to \pi} \frac{e^{\sin x} - 1}{x - \pi} = \lim_{x \to \pi} \frac{e^{\sin x} \cos x}{1} = 1 \cdot (-1) = -1.$$

4. *(12 points)* Determine the absolute extreme values of the function

$$M(x) = x^4 - 32x + 10, \qquad 0 \leqslant x \leqslant 3.$$

Also determine where these extreme values occur. Show sufficient work to justify why your answers give the absolute extreme values.

Absolute maximum: $M(\ 0\ ) = 10$

Absolute minimum: $M(\ 2\ ) = -38$

$M'(x) = 4x^3 - 32 = 4(x^3 - 8)$.

The only critical point is at 2 where $M'(2) = 0$.

Since M is continuous on $[0,3]$ and differentiable on $(0,3)$, it suffices to consider values of M at endpoints and at the critical point:

$M(0) = 10$

$M(2) = -38$

$M(3) = -5$

5. *(12 points)* Evaluate each of the following derivatives.

(a) $\dfrac{d}{dx}\tan(x^2)$ $= 2x\sec^2(x^2)$

(b) $\dfrac{d}{dx}\displaystyle\int_0^x \tan(t^2)\,dt$ $= \tan(x^2)$

(c) $\dfrac{d}{dx}\displaystyle\int_{3x}^0 \tan(t^2)\,dt$ $= -\dfrac{d}{dx}\displaystyle\int_0^{3x}\tan(t^2)\,dt = -3\tan(9x^2)$

6. *(10 points)* A continuous function $f(x)$ is increasing for $x \geqslant 0$. Selected values of this function are listed in the following table.

x	0	2	4	6	8	10
$f(x)$	0	2	6	12	18	22

Using these values, proceed as follows to approximate the integral $\displaystyle\int_0^{10} f(x)\,dx$.

(a) Determine the left Riemann sum of f on the interval $[0, 10]$, using the subdivision of this interval into five subintervals of width $\Delta x = 2$.

$f(0)\Delta x + f(2)\Delta x + f(4)\Delta x + f(6)\Delta x + f(8)\Delta x$

$= (0+2+6+12+18)\cdot 2 = 76.$

(b) Determine the right Riemann sum of f on the interval $[0, 10]$, using the same subdivision as in (a).

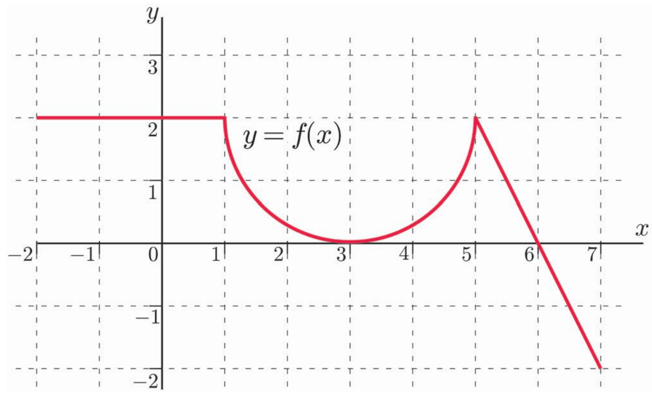
$f(2)\Delta x + f(4)\Delta x + f(6)\Delta x + f(8)\Delta x + f(10)\Delta x$

$= (2+6+12+18+22)\cdot 2 = 120.$

(c) Using the information given, can you be certain that the exact value of the integral lies between the two estimates given in (a) and (b)? Explain briefly.

Yes, we can be sure of this since f is increasing.
For an increasing function, the lower Riemann sum is the left Riemann sum; and the upper Riemann sum is the right Riemann sum.

7. *(10 points)* The graph of a function f, shown on the right, consists of two line segments and a semicircle. *Using familiar geometric formulas for area* (but without explicit formulas for antiderivatives), give exact values for each of the following integrals.



(a) $\int_{-2}^{1} f(x)\,dx = \underbrace{3 \times 2}_{\text{rectangle}} = 6$

(b) $\int_{1}^{5} f(x)\,dx = \underbrace{4 \times 2}_{\text{rectangle}} - \underbrace{\tfrac{1}{2} \times 4\pi}_{\text{semicircle}} = 8 - 2\pi$

(c) $\int_{5}^{6} f(x)\,dx = \underbrace{\tfrac{1}{2} \times 2 \times 1}_{\text{triangle}} = 1$

(d) $\int_{6}^{7} f(x)\,dx = -\underbrace{\tfrac{1}{2} \times 2 \times 1}_{\text{triangle}} = -1$

(e) $\int_{-2}^{7} f(x)\,dx = 6 + 8 - 2\pi + 1 - 1 = 14 - 2\pi$

8. *(12 points)* Evaluate each of the following integrals exactly, using any methods you know (including use of antiderivatives, symmetry, etc.). Simplify your answers.

(a) $\displaystyle\int_{-3}^{3} (\cancel{4x^3} + 6x^2 - \cancel{5x})\, dx = 2\int_0^3 6x^2\,dx = 2\cdot 2x^3 \Big|_0^3 = 108.$

Odd degree terms don't contribute; even degree terms give an even function.

(b) $\displaystyle\int_0^{3\pi} \sin(t)\, dt = -\cos t \Big|_0^{3\pi} = -(-1) - (-1) = 2.$

(c) $\displaystyle\int_0^{3} e^{2x}\, dx = \frac{1}{2} e^{2x} \Big|_0^3 = \frac{1}{2}(e^6 - 1).$

(d) $\displaystyle\int_1^{4} \frac{x+1}{\sqrt{x}}\, dx = \int_1^4 \left(x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right) dx = \left[\frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}\right]_1^4$

$\displaystyle = \left[\frac{2}{3}\cdot 8 + 2\cdot 2\right] - \left[\frac{2}{3} + 2\right] = \frac{20}{3}.$

9. *(8 points)* At time t hours after noon, the outdoor temperature is found to be

$$T(t) = 48 - t^2$$

degrees Fahrenheit. Determine the average temperature for $0 \leqslant t \leqslant 3$, i.e. between noon and 3pm.

$\bar{T} = \frac{1}{3-0} \int_0^3 T(t)\,dt = \frac{1}{3}\left[48t - \frac{t^3}{3}\right]_0^3 = \frac{1}{3}\cdot\left[48\cdot 3 - 9\right] =$ 45 degrees F.

10. *(12 points)* A projectile is forced along a straight track with acceleration $a(t) = 6 - t$ ft/sec^2 for $t \geqslant 0$, where t is the time in seconds. Assume the projectile starts from rest (velocity $v(0) = 0$) at the origin (position $s(0) = 0$).

(a) Determine the velocity $v(t)$ of the projectile in ft/sec.

$$v(t) = \int a(t)\,dt = \int (6-t)\,dt = 6t - \frac{t^2}{2} + C.$$

Since $v(0) = 0$, we have $C = 0$ so $v(t) = 6t - \frac{t^2}{2}$ ft/sec.

(b) Determine the position $s(t)$ of the projectile, in feet.

$$s(t) = \int v(t)\,dt = \int \left(6t - \frac{t^2}{2}\right) dt = 3t^2 - \frac{t^3}{6} + K.$$

Since $s(0) = 0$, $K = 0$ so $s(t) = 3t^2 - \frac{t^3}{6}$ ft.

(c) At which time $t > 0$ does the projectile reverse direction?

From (a), $v(t) = \frac{1}{2}(12 - t)t$.

$v(t) > 0$ for $0 < t < 12$; $v(t) < 0$ for $t > 12$.

The projectile reverses direction at time $t = 12$ sec.

(d) At which time $t > 0$ does the projectile return to the origin?

From (b), $s(t) = \frac{1}{6}(18 - t)t^2$. The projectile returns to the origin at time $t = 18$ sec $(s(18) = 0)$.

(SCRATCH WORK)

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Useful Formulas

$f(u)$	$f'(u)$
$\tan u$	$\sec^2 u$
$\cot u$	$-\csc^2 u$
$\sec u$	$\sec u \tan u$
$\csc u$	$-\csc u \cot u$
$\ln f(u)$	$f'(u)/f(u)$
$\log_b u$	$1/(u \ln b)$
$\sin^{-1} u$	$1/\sqrt{1-u^2}$
$\tan^{-1} u$	$1/(1+u^2)$
$\sec^{-1} u$	$1/(\lvert u\rvert\sqrt{u^2-1})$
$\cos^{-1} u$	$-1/\sqrt{1-u^2}$
$\cot^{-1} u$	$-1/(1+u^2)$
$\csc^{-1} u$	$-1/(\lvert u\rvert\sqrt{u^2-1})$

Chain rule: $(f \circ g)'(x) = f'(g(x))g'(x)$

A circle of radius r has circumference $2\pi r$ and area πr^2.

A triangle of base b and height h has area $\frac{1}{2}bh$.

A trapezoid, with bases b_1, b_2 and height h, has area $\frac{1}{2}(b_1+b_2)h$.

For instructors' use only:

Question	1	2	3	4	5	6	7	8	9	10	Total
Points	8	8	8	12	12	10	10	12	8	12	100
Score											