RECONSTRUCTING PROJECTIVE PLANES FROM SEMIBIPLANES

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Abstract. From a projective plane Π with a homology τ of order 2, one obtains an incidence system having as points and blocks the $\langle\tau\rangle$-orbits of length 2 on the points and lines of Π, and with incidence inherited from Π. The resulting structure, denoted by Π/τ, is an example of a homology semibiplane.

We have shown that a Desarguesian projective plane of odd prime order is uniquely reconstructible from its homology semibiplane (although such a reconstruction is not in general unique for other planes). This is one step towards classifying projective planes of prime order which admit a collineation of order 2.

More generally we reduce the problem of 'lifting' semibiplanes to projective planes, to an equivalent (but better codified) problem in linear algebra. Conceivably this technique may produce new projective planes from the semibiplanes of known planes.

Key words. semibiplane, projective plane, homology

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1. Homology Semibiplanes. A **semibiplane** is an incidence system $\Sigma = (\mathcal{P}, \mathcal{B})$ consisting of a set $\mathcal{P}$ of **points**, and a set $\mathcal{B}$ consisting of certain subsets of $\mathcal{P}$ called **blocks**, such that

(i) any two distinct points lie in either 0 or 2 common blocks;

(ii) any two distinct blocks meet in either 0 or 2 points;

(iii) Σ is connected; and

(iv) every block contains at least 3 points.

(See Hughes [5].) It is easy to show that given any semibiplane $\Sigma = (\mathcal{P}, \mathcal{B})$, there exist integers v, k such that $|\mathcal{P}| = |\mathcal{B}| = v$, each block contains exactly k points, and each point belongs to exactly k blocks.

EXAMPLE 1.1. Take vertices of the regular icosahedron as points, and the regular pentagons among its edge circuits as blocks. This forms a semibiplane with $v = 12$, $k = 5$.

Two blocks $L, L' \in \mathcal{B}$ are **parallel** if $L = L'$ or $L \cap L' = \emptyset$. Each block is parallel to exactly $t = v - \binom{k}{2}$ blocks. If parallelism is an equivalence relation on the blocks, then the dual relation on the points ($P \sim P' \iff P = P'$ or P, P' are unjoined) is also an equivalence relation. In this case the semibiplane Σ is said to be **divisible**, and each class (of points or of blocks) has size $t = v - \binom{k}{2}$. (Observe that Example 1.1 is divisible

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with $t = 2$. Antipodal vertices are unjoined points; pentagons lying in parallel planes are parallel blocks.)

Suppose now that Π is a projective plane of order n, admitting an **involutory homology** τ (so that n is odd). That is, τ is a collineation of order 2, fixing pointwise some line of Π and an additional point of Π. (Our terminology concerning projective planes is standard and follows [6].) As in [5], we may construct a divisible semibiplane with $v = \frac{1}{2}(n^2 - 1)$, $k = n$, $t = \frac{1}{2}(n-1)$ as follows: 'points' and 'blocks' are the $\langle\tau\rangle$-orbits of length 2 on the points and lines of Π, and incidence is inherited from Π. We denote the resulting semibiplane by Π/τ. This motivates the definition: a **homology semibiplane** is a divisible semibiplane with $t = \frac{1}{2}(k-1)$ (i.e. k is odd, $v = \frac{1}{2}(k^2 - 1)$). Its **order** is k. This suggests the following

PROBLEM 1.2. Given a homology semibiplane Σ, is $\Sigma \cong \Pi/\tau$ for some Π,τ? Moreover, how many *nonequivalent* pairs (Π, τ) yield Σ in this way?

(We say that the pairs (Π_1, τ_1), (Π_2, τ_2) are **equivalent** if there exists an isomorphism $\psi : \Pi_1 \to \Pi_2$ such that $\psi \circ \tau_1 = \tau_2 \circ \psi$; clearly in this case $\Pi_1/\tau_1 \cong \Pi_2/\tau_2$.) In [11] we prove

THEOREM 1.3. *If $\Pi/\tau \cong \Pi'/\tau'$ where Π' is Desarguesian of odd prime order, then $\Pi \cong \Pi'$.*

Under the broader hypothesis '...odd prime *power* order', we failed to obtain the same conclusion (the 'linear algebra' becomes much more involved in this case); see remarks in Section 2 concerning the case $n = 9$.

We indicate in Section 2 how these methods apply to the problem of classifying projective planes with involutory homologies. Section 3 describes the key idea in the proof of 1.3, namely the reduction of Problem 1.2 to a linear algebra problem. In Section 4 we remark on the situation for involutory collineations *other than* homologies. And Section 5 places our Problem 1.2 in the context of a more general 'lifting' problem.

2. Classifying planes with involutory homologies. We point out that Problem 1.2 is a subproblem of

PROBLEM 2.1. Given an odd integer n, how many (isomorphism classes of) projective planes of order n exist, admitting an involutory homology?

To solve 2.1 for a particular value of n, one proceeds in two steps:

(i) First find all homology semibiplanes of order n (up to isomorphism). Then
(ii) 'lift' each such semibiplane, in as many (essentially different) ways as possible, to obtain projective planes (this is Problem 1.2).

For instance Matulić-Bedenić [10] shows the uniqueness of a projective plane of order 11 admitting an involutory homology, by first proving the uniqueness of the homology semibiplane Σ of order 11, then showing that Σ lifts uniquely. Note that Theorem 1.3 generalizes step (ii) of this argument. To attempt to prove that all projective planes of prime order admitting an involutory collineation are Desarguesian, amounts to generalizing step (i); however this is evidently too difficult for current methods.

Also interesting is the case $n = 9$ treated in [7], where Janko and van Trung show that

(i) there are precisely three homology semibiplanes of order 9, and

(ii) they lift as follows:

$\Sigma_1 \longrightarrow$ Desarguesian plane PG(2,9)

$\Sigma_2 \longrightarrow$ Hughes plane of order 9

$\Sigma_3 \nearrow$ Hall plane of order 9

$\Sigma_3 \searrow$ dual Hall plane of order 9

(In particular the only planes of order 9 admitting involutory homologies are the four known planes of order 9.) The example of Σ_1 above suggests that Theorem 1.3 might possibly extend to any Desarguesian plane Π', although not to an arbitrary plane Π' (in view of the case Σ_3 above).

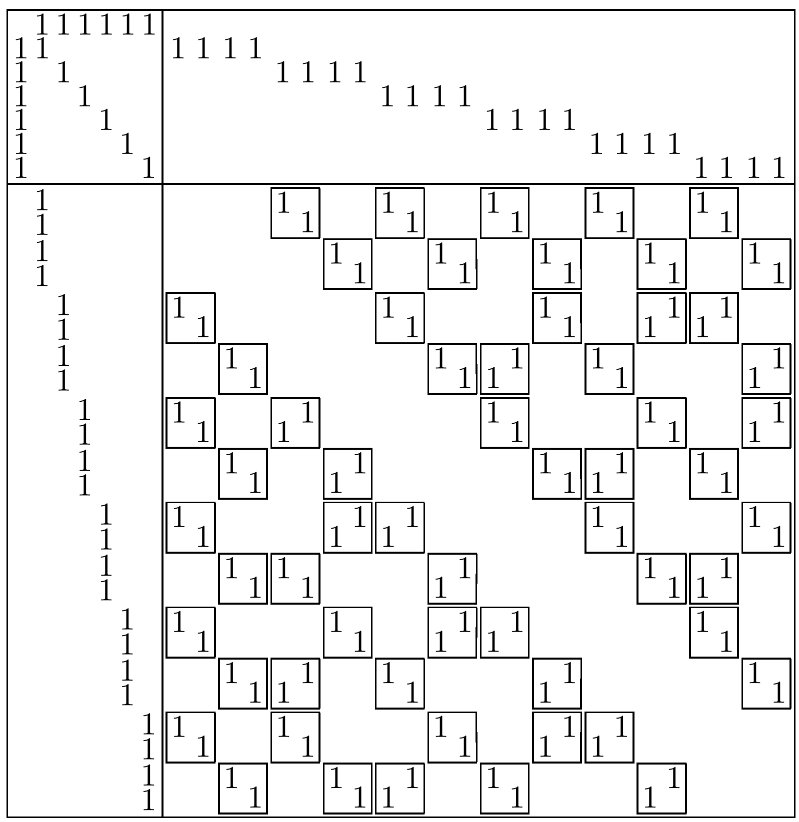

FIGURE 3.1. Incidence matrix of $\Pi' = \mathrm{PG}(2,5)$

3. 'Linearizing' the lifting problem. Let us see how Problem 1.2 reduces to a linear problem. The general case is formally treated in [11], and here we illustrate the key idea with particular reference to the case of $\Pi' = \mathrm{PG}(2,5)$, the unique (Desarguesian) projective plane of order 5. The incidence matrix of Π', as shown in Figure 3.1, makes evident the action of an involutory homology τ' which fixes the seven points (resp., lines) corresponding to the first seven rows (resp., columns) of the matrix, and permuting the remaining points (resp., lines) in pairs corresponding to adjacent rows (resp., columns). From the bottom-right 24×24 submatrix thereof we obtain the incidence matrix of the resulting homology semibiplane $\Sigma' = \Pi'/\tau'$, shown as Figure 3.2. It turns out that Σ' is isomorphic to Example 1.1, this being in fact the unique homology semibiplane of order 5. (*Aside.* Using the construction of Σ' from Π' we see that $\mathrm{Aut}\,\Sigma' \supseteq \mathrm{C}_{\mathrm{Aut}\,\Pi'}(\tau')/\langle\tau'\rangle \cong \mathrm{GL}(2,5)/Z_2$ where $Z_2 \subset \mathrm{Z}(\mathrm{GL}(2,5))$ is of order 2; cf. [5]. It may in fact be shown that $\mathrm{Aut}\,\Sigma' \cong \mathrm{GL}(2,5)/Z_2$. Meanwhile from the earlier construction it is evident that $\mathrm{Aut}\,\Sigma'$ contains the isometry group of the icosahedron, namely $\mathrm{A}_5 \times \mathrm{Z}_2$, which is a subgroup of index 2 in $\mathrm{GL}(2,5)/Z_2$.)

FIGURE 3.2. Incidence matrix of the homology semibiplane Σ'

$$\begin{array}{|cc|cc|cc|cc|cc|cc|}
\hline
 & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & & & 1 & & 1 & & 1 & & 1 & & 1 \\
\hline
1 & & & & 1 & & & 1 & & 1 & 1 & \\
 & 1 & & & & 1 & 1 & & 1 & & & 1 \\
\hline
1 & & 1 & & & & 1 & & & 1 & & 1 \\
 & 1 & & 1 & & & & 1 & 1 & & 1 & \\
\hline
1 & & & 1 & 1 & & & & 1 & & & 1 \\
 & 1 & 1 & & & 1 & & & & 1 & 1 & \\
\hline
1 & & & 1 & & 1 & 1 & & & & 1 & \\
 & 1 & 1 & & 1 & & & 1 & & & & 1 \\
\hline
1 & & 1 & & & 1 & & 1 & 1 & & & \\
 & 1 & & 1 & 1 & & 1 & & & 1 & & \\
\hline
\end{array}$$

Consider now the reverse problem of trying to lift Σ' to a projective plane Π of order 5. To obtain (a 24×24 submatrix of) an incidence matrix for Π, each '1' in the incidence matrix of Σ' (see Fig. 3.2) must be replaced by either $\binom{1\,0}{0\,1}$ or $\binom{0\,1}{1\,0}$. Furthermore it is clear that the four 1's corresponding to any digon $(P,Q;L,M)$ of Σ' must be replaced by three $\binom{1\,0}{0\,1}$'s and one $\binom{0\,1}{1\,0}$, or by one $\binom{1\,0}{0\,1}$ and three $\binom{0\,1}{1\,0}$'s. (We call $(P,Q;L,M)$ a **digon** of Σ' if $P \neq Q$ are points of Σ', joined by the blocks $L \neq M$.)

Now let $\mathcal{F} \subset \mathcal{P} \times \mathcal{B}$ be the set of **flags** of Σ', i.e. the set of pairs (P,L) such that P is a point contained in the block L. Let $F = \mathrm{GF}(2)$, the field of order 2. The above remarks show firstly that any plane Π obtained from Σ' is determined by a corresponding function $\alpha : \mathcal{F} \to F$, namely $\alpha(P,L) = 0$ or 1 according as the (P,L)-entry of the incidence matrix of Σ' is replaced by $\binom{1\,0}{0\,1}$ or $\binom{0\,1}{1\,0}$; and secondly that

$$\alpha(P,L) + \alpha(P,M) + \alpha(Q,L) + \alpha(Q,M) = 1$$

for any digon $(P,Q;L,M)$ of Σ'.

Let V be the vector space of all functions $\mathcal{F} \to F$, with standard basis $\{\chi_{P,L} : (P,L) \in \mathcal{F}\}$, where

$$\chi_{P,L}(Q,M) = \begin{cases} 1, & \text{if } (Q,M) = (P,L); \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\dim V = |\mathcal{F}| = 60$ (or in general $\frac{1}{2}k(k^2-1)$). Define a bilinear form on V by

$$(\beta,\gamma) = \sum_{(P,L)\in\mathcal{F}} \beta(P,L)\gamma(P,L), \qquad \text{for } \beta,\gamma \in V.$$

Then we require that $(\alpha, \delta) = 1$ for all $\delta \in \mathcal{D}$, where

$$\mathcal{D} = \{\chi_{P,L} + \chi_{P,M} + \chi_{Q,L} + \chi_{Q,M} : (P, Q; L, M) \text{ is a digon in } \Sigma'\}.$$

In particular

$$\alpha \in \langle \delta + \delta' : \delta, \delta' \in \mathcal{D} \rangle^{\perp}.$$

This is now a linear condition on α, which provides the basic key to limiting the possibilities for Π. Note however that distinct α's may yield isomorphic planes Π. In [11] we consider how to suppress these redundant solutions for α in the general case of Problem 1.2, as well as prove Theorem 1.3 in case Π' is Desarguesian of odd prime order.

4. Elation and Baer semibiplanes. An **elation semibiplane** is a divisible semibiplane with $t = k/2$ (i.e. k is even, $v = k^2/2$). Examples with $k = n$ may be constructed (cf. [5]) from projective planes of even order n with involutory elations; however other constructions are known. Indeed Jungnickel [9] (and implicitly Drake [2]) construct elation semibiplanes with $k = 2q$, q an odd prime power, using generalized Hadamard matrix constructions [8] (for q an odd prime the latter matrix constructions are due to Butson [1]). This family of elation semibiplanes was independently constructed by Wild [12]. By [3], none of the semibiplanes in this family 'lift' to projective planes with involutory elations. We wish to thank Dieter Jungnickel for bringing these examples to our attention. This presents a contrast with the situation for homology semibiplanes, where all constructions known to us arise from projective planes with involutory homologies.

As with homology semibiplanes, the problem of lifting elation semibiplanes (also 'Baer' semibiplanes; cf. [5]) reduces to a problem in linear algebra by the same method of Section 3.

5. Classifying planes with given groups. Our interest in lifting semibiplanes stems from a consideration of the more general

PROBLEM 5.1. Given an abstract finite group G and integer $n > 1$, classify (up to equivalence) all projective planes of order n which admit G as a collineation group.

Each solution to this problem may be expressed in the form of a matrix $\mathcal{A}$, a sort of *generalized incidence matrix* with entries in the group algebra $\mathbb{Q}G$ over the rational field $\mathbb{Q}$, as proposed by Hughes [3], [4], who showed that such matrices satisfy certain relations. If $P_1, P_2, \ldots, P_w$ and $L_1, L_2, \ldots, L_w$ are representatives of the distinct point and line orbits of G, then by definition the (i, j)-entry of $\mathcal{A}$ is

$$\sum\{g \in G : P_i^g \in L_j\}.$$

Another $w \times w$ matrix A (with integer entries) is defined as the homomorphic image of $\mathcal{A}$ under the homomorphism $\mathbb{Q}G \to \mathbb{Q}$, $\sum a_g g \mapsto \sum a_g$; i.e. the (i, j)-entry of A is $|\{g \in G : P_i^g \in L_j\}|$.

To illustrate we return to the example of Section 3 in which $G = \langle \tau' \rangle$, τ' an involutory homology of $\Pi' = \mathrm{PG}(2,5)$. In this case the matrix $\mathcal{A}$ is given by Figure 5.2, in which $\gamma = 1 + \tau' \in \mathbb{Q}G$. To obtain A from $\mathcal{A}$ simply replace each τ' entry by 1, and each γ entry by 2. The relations of [3], [4] amount to

$$\mathcal{A}D^{-1}\mathcal{A}^{\mathrm{T}} = \mathcal{A}^{\mathrm{T}}D^{-1}\mathcal{A} = 5\mathcal{D} + \gamma\mathrm{J}, \qquad AD^{-1}A^{\mathrm{T}} = A^{\mathrm{T}}D^{-1}A = 5D + 2\mathrm{J},$$

where $\mathcal{D} = \begin{pmatrix} \gamma\mathrm{I}_7 & 0 \\ 0 & \mathrm{I}_{12} \end{pmatrix}$, $D = \begin{pmatrix} 2\mathrm{I}_7 & 0 \\ 0 & \mathrm{I}_{12} \end{pmatrix}$, I_r = identity matrix of order r, and J is the 19×19 matrix whose every entry is 1.

$$\left[\begin{array}{ccccccc|cc|cc|cc|cc|cc|cc}
 & \gamma & \gamma & \gamma & \gamma & \gamma & \gamma & & & & & & & & & & & & \\
\gamma & \gamma & & & & & & \gamma & \gamma & & & & & & & & & & \\
\gamma & & \gamma & & & & & & & \gamma & \gamma & & & & & & & & \\
\gamma & & & \gamma & & & & & & & & \gamma & \gamma & & & & & & \\
\gamma & & & & \gamma & & & & & & & & & \gamma & \gamma & & & & \\
\gamma & & & & & \gamma & & & & & & & & & & \gamma & \gamma & & \\
\gamma & & & & & & \gamma & & & & & & & & & & & \gamma & \gamma \\
\hline
 & \gamma & & & & & & & & 1 & & 1 & & 1 & & 1 & & 1 & \\
 & \gamma & & & & & & & & & 1 & & 1 & & 1 & & 1 & & 1 \\
\hline
 & & \gamma & & & & & 1 & & & & 1 & & & 1 & & \tau' & \tau' & \\
 & & \gamma & & & & & & 1 & & & & 1 & \tau' & & 1 & & & \tau' \\
\hline
 & & & \gamma & & & & 1 & & \tau' & & & & 1 & & & 1 & & \tau' \\
 & & & \gamma & & & & & 1 & & \tau' & & & & 1 & \tau' & & 1 & \\
\hline
 & & & & \gamma & & & 1 & & & \tau' & \tau' & & & & 1 & & & 1 \\
 & & & & \gamma & & & & 1 & 1 & & & \tau' & & & & 1 & \tau' & \\
\hline
 & & & & & \gamma & & 1 & & & 1 & & \tau' & \tau' & & & & 1 & \\
 & & & & & \gamma & & & 1 & \tau' & & 1 & & & \tau' & & & & 1 \\
\hline
 & & & & & & \gamma & 1 & & 1 & & & 1 & & \tau' & \tau' & & & \\
 & & & & & & \gamma & & 1 & & 1 & \tau' & & 1 & & & \tau' & &
\end{array}\right]$$

FIGURE 5.2. Generalized incidence matrix $\mathcal{A}$ for the action of an involutory homology τ' on $\Pi' = \mathrm{PG}(2{,}5)$

To solve 5.1 in general,

(i) first determine all possibilities for A, using the necessary relations for A (as in [3], [4]) and various arguments based on the structure of G. Then

(ii) 'lift' each such A to as many $\mathcal{A}$'s as possible.

Step (ii) is very difficult in general, although for $|G| = 2$ we have seen that step (ii) reduces to a linear problem, which is much more tractable. Returning to our example, the lifting of A to $\mathcal{A}$ is accomplished by our function α. Namely, the bottom-right 12×12 submatrix of A is identical to Figure 3.2. An entry '1' thereof, say in the (P, L)-position where (P, L) is a flag of Σ', lifts to an entry '1' or 'τ'' of $\mathcal{A}$ according as $\alpha(P, L) = 0$ or 1 (cf. Figure 3.1).

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