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PSL$(3,q)$ AND PSU$(3,q)$ ON PROJECTIVE PLANES OF ORDER q^4

Abstract. Let $q = p^m$ be an odd prime power.

We show that a projective plane Π of order q^4 admitting a collineation group $G \cong$ PSL$(3,q)$ or PSU$(3,q)$, has a G-invariant Desarguesian subplane Π_0 of order q or q^2 respectively, and that G contains involutory homologies of Π (with possible exceptions for $q = 3$, 5 or 11).

We also show that a projective plane Π of order q^2 admitting a collineation group $G \cong$ PSL$(2,q)$ or PGL$(2,q)$, has a G-invariant $(q+1)$-arc or dual thereof, for most reasonably small odd q.

Most of our tools and techniques are known, except seemingly for our results concerning an abelian planar collineation group P of a projective plane Π. These results are applied here in each of the above situations for P a Sylow p-subgroup of G, and presumably they will enjoy broader application.

1. Results

Let $q = p^m$ be a prime power, $m \geq 1$, throughout. It is well known that a projective plane which admits $G \cong$ PSL$(3,q)$ as a collineation group is necessarily Desarguesian. (Indeed, a Sylow p-subgroup of G suffices; see Dembowski [4]). For planes of order q^2 or q^3, the following characterizations 1.1,2 are known.

1.1 THEOREM (Unkelbach, Dembowski, Lüneburg). *If Π is a projective plane of order q^2 admitting a collineation group $G \cong$ PSL$(3,q)$, then Π is either Desarguesian or a generalized Hughes plane. Conversely, any Desarguesian or generalized Hughes plane of order q^2, save the exceptional Hughes plane of order 7^2, admits PSL$(3,q)$ as a collineation group.*

(For completeness, we indicate a proof of 1.1 in §4, using the results of Unkelbach [35], Dembowski [6] and Lüneburg [24].) The *generalized Hughes planes* include the infinite

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family of *(ordinary) Hughes planes* (one such plane of order q^2 for each odd prime power q), together with the seven *exceptional Hughes planes* having order 5^2, 7^2, 11^2, 11^2, 23^2, 29^2, 59^2 respectively (see [24]).

1.2 THEOREM (Dempwolff [7]). *If* Π *is a projective plane of order* q^3 *admitting* $G \cong$ PSL$(3,q)$, *then* Π *has a* G*-invariant Desarguesian subplane* Π_0 *of order* q *on which* G *acts faithfully, and* G *contains elations and involutory homologies of* Π. *(Some additional orbit information is obtained in [7].)*

The only known occurrences of 1.2 are the Desarguesian planes and the Figueroa planes [10], [14]. A comparison of the above results shows that as the order of Π is increased relative to $|G|$, it becomes increasingly difficult to completely classify the possibilities for Π to within isomorphism. We go one step further by proving (in §9) the following.

1.3 THEOREM. *Suppose that* Π *is a projective plane of order* q^4 *admitting* $G \cong$ PSL$(3,q)$, q *odd. If* $q > 3$ *then the following must hold.*

(i) G *leaves invariant a Desarguesian subplane* Π_0 *of order* q*, on which* G *induces the little projective group.*

(ii) The involutions in G *are homologies of* Π*, and those elements of* G *which induce elations of* Π_0 *are elations of* Π.

If $q = 3$ *then the same two conclusions must hold, under the additional hypothesis that* G *acts irreducibly on* Π.

(A collineation group is **irreducible** if it leaves invariant no point, line or triangle.) In the same way we try to extend the following well-known result to planes of larger order.

1.4 THEOREM (Hoffer [16]). *Suppose that* Π *is a projective plane of order* q^2 *admitting a collineation group* $G \cong$ PSU$(3,q)$. *Then* Π *is Desarguesian and there is (to within equivalence) a unique faithful action of* G *on* Π. G *commutes with* δ *for some hermitian polarity* δ *of* Π*, and so* G *leaves invariant the corresponding hermitian unital.*

(More is said about hermitian unitals in §5.) The following extension is proven (together with Theorem 1.3) in §9.

1.5 THEOREM. *Suppose that* Π *is a projective plane of order* q^4 *admitting* $G \cong \mathrm{PSU}(3,q)$, q *odd.*

(a) If $q \neq 5, 11$ *then the following must hold:*

(i) G *leaves invariant a Desarguesian subplane* Π_0 *of order* q^2, *on which* G *acts faithfully, leaving invariant a hermitian unital.*

(ii) The involutions in G *are homologies of* Π.

(b) If $q = 5$ *or* 11 *and either conclusion (i) or (ii) above fails, then* Π *or its dual has a point orbit* $\mathcal{O}$ *of length* q^3+1, *such that* $\mathcal{O}$ *is an arc (for* $q = 5$ *or* 11*) or a hermitian unital embedded in* Π *(for* $q = 5$ *only).*

(c) If the hypothesis $G \cong \mathrm{PSU}(3,q)$ *is replaced by* $G \cong \mathrm{PGU}(3,q)$ *then (i), (ii) must hold for all odd prime powers* q, *including* $5, 11$.

In Theorems 1.3,5 it is clear that any G-invariant subplane of Π contains Π_0 (whenever Π_0 itself exists) since Π_0 is generated by the centres of involutory homologies in G.

The only known occurrences of 1.3,5 are Desarguesian and Hughes planes. No analogues of Theorems 1.3,5 are known for q even. Indeed, 1.3 fails for $q = 2$. Namely, if Π or its dual is a Lorimer-Rahilly translation plane of order 16 (see [22]) then Π admits a collineation group $G \cong \mathrm{PSL}(3,2)$ such that $\mathrm{Fix}(G)$ is a subplane of order 2. We remark on the situation for $q = 3$ in §10, pointing out an intriguing similarity with the case $q = 2$.

If $G \cong \mathrm{PSL}(3,q)$ or $\mathrm{PSU}(3,q)$ and $\tau \in G$ is an involution, then $\mathrm{C}_G(\tau)'/\langle\tau\rangle \cong \mathrm{PSL}(2,q)$. Accordingly in proving 1.3,5, in case τ is a Baer involution of Π, we require results concerning the action of $\mathrm{PSL}(2,q)$ on a plane of order q^2. We prove some such results, which are new and interesting in their own right. First, however, we make extensive use of the following well known result, proven in [25].

1.6 THEOREM (Lüneburg, Yaqub). *Suppose that* Π *is a projective plane of order* q *admitting* $G \cong \mathrm{PSL}(2,q)$. *Then* Π *is Desarguesian.* G *acts irreducibly on* Π *for odd* $q > 3$, *and leaves invariant a triangle but no point or line for* $q=3$. G *fixes a point and/or line of* Π *if* q *is even.*

In [28] we also proved the following.

1.7 THEOREM. *Suppose that* Π *is a projective plane of order* q^2 *admitting* $G \cong \mathrm{PSL}(2,q)$, q *odd. Then one of the following must hold:*

(i) G *acts irreducibly on* Π;

(ii) $q=3$ *and* G *fixes a triangle but no point or line of* Π;

(iii) $q=5$, $\mathrm{Fix}(G)$ *consists of an antiflag* (X,l), *and* G *has point orbits of length 5, 5, 6, 10 on* l; *or*

(iv) $q=9$ *and* $\mathrm{Fix}(G)$ *consists of a flag.*

For certain values of q we shall make use of the following result, proven in §7. Note that the case $G/K \cong \mathrm{PGL}(2,q)$ is especially included.

1.8 THEOREM. *Suppose* $G=\mathrm{GL}(2,q)$ *acts as a group of collineations of a projective plane* Π *of order* q^2, q *odd, such that the kernel* K *of this action satisfies* $K \leq \mathrm{Z}(G)$, $2 \bigm| |K|$. *Then* G *fixes no point or line of* Π, *and leaves invariant a triangle precisely when* $q=3$. *Furthermore, one of the following must hold:*

(i) there is a point orbit which is a $(q+1)$*-arc;*

(ii) the dual of (i); or

(iii) $q>10^6$ *and* q *is a square.*

In §8 we prove the following related result, although this is only required for $q=5$, 17 in proving Theorem 1.5.

1.9 THEOREM. *Suppose that* Π *is a projective plane of order* q^2 *admitting* $G \cong \mathrm{PSL}(2,q)$, q *odd, and that* q *is not a square (i.e.* m *is odd). If* $q \neq 5$ *then one of the following must hold:*

(i) there is a point orbit which is a $(q+1)$*-arc;*

(ii) the dual of (i); or

(iii) $q > 5000$ *and* $q \equiv 3 \mod 8$.

Furthermore if $q = 5$ *then either (i) or (ii) must hold under the additional hypothesis that* G *acts irreducibly on* Π.

In the situation of Theorems 1.8,9 it is natural to conjecture that conclusions (i),(ii) must hold in all cases, that such a $(q+1)$-arc generates a proper subplane of Π; and that G contains involutory homologies. These statements we could not verify (see however Corollary 5.2 of [28]).

This paper is a sequel to [28], which we will quote freely. Most of the new results contained herein were contained in the author's doctoral thesis [27] under the kind supervision of Professor Chat Y. Ho.

2. Notation and Preliminaries

Most of our notation and terminology is standard. Some better-known results are stated below without proof and the reader is referred to [11] for group theory, and [5] or [18] for projective planes.

We denote the cyclic group of order n by C_n, and the symmetric and alternating groups of degree n by S_n and A_n. For a finite group G we denote by G' the derived subgroup of G, and by $\mathrm{Syl}_p(G)$ the class of all Sylow p-subgroups of G. We denote by $G \rtimes H$ the semidirect product of G with H (see [31]). An **involution** is a group element of order 2.

For a permutation group $G \leq \mathrm{Sym}\,\Omega$ and an element $X \in \Omega$, we denote the **stabilizer** of X by $G_X = \{g \in G : X^g = X\}$, and the G-**orbit** of X by $X^G = \{X^g : g \in G\}$. We say that G acts **semiregularly** on Ω if $G_X = 1$ for all $X \in \Omega$. Also G acts **regularly** if it acts both transitively and semiregularly.

Let Π be a finite projective plane. A pair (X, l) consisting f a point X and a line l of Π, is a **flag** or an **antiflag** according as $X \in l$ or $X \notin l$. A collineation $g \neq 1$ of Π

is a **generalized** (X,l)**-perspectivity** if it fixes the point X and the line l, and if any additional fixed points (resp., lines) lie on l (resp., pass through X). If g fixes l pointwise and X linewise, g is an (X,l)-**perspectivity** with **centre** X and **axis** l. (Following [5], we include the identity $1 \in \mathrm{Aut}\,\Pi$ as both a perspectivity and a generalized perspectivity.) We say **elation** or **homology** in place of 'perspectivity' according as $X \in l$ or $X \notin l$. If S is a set of collineations of Π then by $\mathrm{Fix}(S)$ we mean the full closed substructure of Π consisting of all points and lines fixed by every element of S.

2.1 PROPOSITION. *If G acts on a projective plane Π such that $\mathrm{Fix}(G) = \emptyset$, then for any $N \trianglelefteq G$, $\mathrm{Fix}(N)$ is either empty, a triangle, or a (not necessarily proper) subplane of Π.*

For a proof, see [13,Cor.3.6].

2.2 THEOREM (Bruck). *If Π_0 is a proper subplane of Π, then their respective orders n_0, n satisfy either $n_0^2 = n$ or $n_0^2 + n_0 \le n$.*

If $n_0 = n$, we call Π_0 a **Baer** subplane of Π. In this case each point of Π lies on some line of Π_0. If $\mathrm{Fix}(G)$ is a subplane (respectively, a Baer subplane, a triangle) of Π, we say that G is a **planar** (resp., **Baer**, **triangular**) collineation group of Π. A **quasiperspectivity** is either a perspectivity or a Baer collineation.

2.3 THEOREM (Roth). *In 2.2 if we assume in addition that $\Pi_0 = \mathrm{Fix}(G)$ for some collineation group G of Π, then either $n_0^2 = n$ or $n_0^2 + n_0 + 2 \le n$.*

2.4 PROPOSITION. *If Π is a finite projective plane with Baer collineation group G, then $|G| \,\big|\, n(n-1)$.*

Proposition 2.4, together with its analogue for perspectivities (see Lemma 4.10 of [18]) are elementary and will often be used without explicit mention.

2.5 THEOREM (Baer). *Any involutory collineation of a finite projective plane is a quasi-perspectivity.*

2.6 PROPOSITION. *If g_i is an (X_i, l_i)-perspectivity of Π, $i = 1, 2$, $X_1 \neq X_2$, $l_1 \neq l_2$ then g_1g_2 is a generalized $(l_1 \cap l_2, X_1X_2)$-perspectivity of Π.*

3. The Groups PSL$(2, q)$, PGL$(2, q)$

We assume the reader's familiarity with the classification of subgroups of PSL$(2, q)$ as given in [9], [19] or [34]. Note that if $G \cong$ PGL$(2, q)$ then G is isomorphic to a subgroup of PSL$(2, q^2)$, so by applying the latter classification to PSL$(2, q^2)$ as well as to $G' \cong$ PSL$(2, q)$, we may in fact classify the subgroups of G.

3.1 LEMMA. *If $q = p^m$ is odd then*

(i) PGL$(2, q)$ *has a single conjugacy class of elements of order p;*

(ii) PSL$(2, q)$ *has exactly 2 conjugacy classes of elements of order p; it has 2 or 1 conjugacy class(es) of subgroups of order p according as q is or is not a square.*

Proof of (ii). Let $G \cong$ PSL$(2, q)$, $P \in \mathrm{Syl}_p(G)$, $N = \mathrm{N}_G(P)$, $g \in P \setminus 1$. Then G contains $q^2 - 1$ elements of order p, of which $q - 1$ lie in each of the $q + 1$ conjugates of P; $|G| = \frac{1}{2}q(q^2 - 1)$, $\mathrm{C}_G(g) = P$, $|\mathrm{N}_G(\langle g\rangle)| = (m, 2)q(p - 1)/2$ and the statement follows. □

3.2 LEMMA. *Let $G \cong$ PGL$(2, q)$, q odd. If e is an even divisor of $q + 1$ such that $e > 2$, then G contains a pair of elements x, y of order e such that $\langle x, y\rangle \supseteq G' \cong$ PSL$(2, q)$.*

Proof. If $e = q + 1$ then we may choose x, $y \in G$ of order e such that $\langle x\rangle \neq \langle y\rangle$, and then the classification of subgroups of G (see [9], [19], [34]) gives $\langle x, y\rangle = G$. In particular, we may assume that $q \neq 3, 5, 9$.

Let u, v be the elements of G represented by

$$\begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ \zeta^3 & 0 \end{pmatrix}$$

respectively, where ζ is a generator of the multiplicative group $\mathrm{GF}(q) \setminus \{0\}$. Then $\langle u, v\rangle$ is dihedral of order $q-1$. Now $\mathrm{C}_G(u)$, $\mathrm{C}_G(v)$ are dihedral of order $2(q+1)$ since ζ, ζ^3 are non-squares. We may therefore choose $x \in \mathrm{C}_G(u)$, $y \in \mathrm{C}_G(v)$ of order e.

Now $\langle x, y\rangle \supseteq \langle u, v\rangle$, but $\langle x, y\rangle \not\subseteq \mathrm{N}_G(\langle u, v\rangle)$, since for $q > 3$, the latter is dihedral of order e. If $q \geq 11$ then $\langle x, y\rangle \supseteq G'$ by the classification of subgroups of G. By the initial argument, the only case left to consider is $q = 7, e = 4$, in which case $\langle u, v\rangle \cong \mathrm{S}_3$, $\langle x, y\rangle \supseteq G'$ unless $\langle x, y\rangle \cong \mathrm{S}_4$. But in the latter case $u = x^2$, $v = y^2$ generate an elementary abelian group of order 4, a contradiction. □

The following is proven in [28].

3.3 THEOREM. *If a projective plane* Π *of order* $n < q$ *admits a collineation group* $G \cong \mathrm{PSL}(2, q)$, *then* Π *is Desarguesian and* $(n, q) = (2, 3)$, *(2,7), (4,5), (4,7) or (4,9). Moreover each of these exceptional cases indeed occurs.*

4. The Groups PSL(3, q), PGL(3, q)

Let $G \cong \mathrm{PSL}(3, q)$, $F = \mathrm{GF}(q)$, $F^\times = F \setminus \{0\}$ so that $|G| = q^3(q^3-1)(q^2-1)/\mu$ where $\mu = (q-1, 3)$. Throughout §4 we assume that $q = p^m$ is odd. Of the following facts concerning G, those which are stated without proof are either well-known or follow by elementary methods from the list in [26] of maximal subgroups of G. (The corresponding list for q even is given in [12]. Note that certain of the following, eg. 4.1(i), fail for q even.)

Consider the following elements and subgroups of G, as represented by matrices in $\mathrm{SL}(3,q)$:

$$\tau = \mathrm{diag}(-1,-1,1), \qquad Z_\tau = \mathrm{Z}\big(\mathrm{C}_G(\tau)\big) = \big\{\mathrm{diag}(d,d,d^{-2}) : d \in F^\times\big\},$$

$$\mathrm{C}_G(\tau) = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e^{-1} \end{pmatrix} : \begin{array}{c} a,b,c,d \in F, \\ e = ad-bc \neq 0 \end{array} \right\},$$

$$\mathrm{C}_G(\tau)' = \left\{ \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} : \begin{array}{c} a,b,c,d \in F, \\ ad-bc = 1 \end{array} \right\} \cong \mathrm{SL}(2,q),$$

$$P_0 = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a,b \in F \right\}, \qquad P_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & 1 \end{pmatrix} : a,b \in F \right\},$$

$$Q = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a,b,c \in F \right\} \in \mathrm{Syl}_p(G),$$

$$P = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : a \in F \right\} = \mathrm{Z}(Q) = Q',$$

$$\mathrm{N}_G(Q) = Q \rtimes K, \qquad K = \big\{\mathrm{diag}(d,e,(de)^{-1}) : d,e \in F^\times\big\}.$$

Likewise define $Z_\omega = \mathrm{Z}\big(\mathrm{C}_G(\omega)\big)$ for any involution $\omega \in G$.

4.1 LEMMA.

(i) *G has a single conjugacy class of involutions, and G acts transitively by conjugation on the set of ordered pairs of commuting distinct involutions. One such pair is $\{\tau, \tau'\}$ where $\tau' = \mathrm{diag}(-1,1,-1)$.*

(ii) $|\mathrm{C}_G(\tau)| = q(q+1)(q-1)^2/\mu, \quad |Z_\tau| = (q-1)/\mu$.

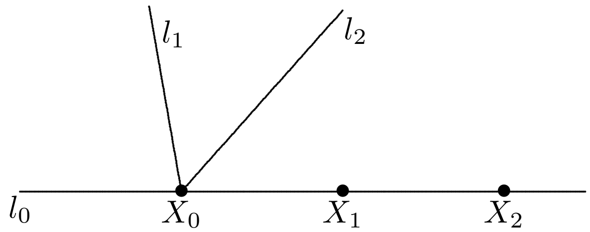
(iii) $\mathrm{C}_G(\tau) = \mathrm{C}_G(\tau)' \rtimes Z_{\tau'}$.

(iv) $\mathrm{C}_G(\tau)/Z_\tau \cong \mathrm{PGL}(2,q)$.

(v) $\mathrm{C}_G(\tau) \cong H/\Xi_\mu$ *where* $H \cong \mathrm{GL}(2,q)$, $\Xi_\mu \le \mathrm{Z}(H)$, $|\Xi_\mu| = \mu$.

(vi) *G contains involutions* τ_1, τ_2 *such that* $\mathrm{C}_G(\tau_1)' \cap \mathrm{C}_G(\tau_2)' \neq 1$, $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)' \rangle = G$.

FIGURE 4A



Proof of (vi). Let Σ be a Desarguesian plane of order q admitting G as its little projective group, and consider the configuration in Σ shown in Figure 4A. Let $\tau_i \in G$ be the involutory (X_i, l_i)-homology of Σ, $i = 1, 2$. Then $\mathrm{C}_G(\tau_1)' \cap \mathrm{C}_G(\tau_2)' \supseteq G(X_0, l_0)$ and $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)' \rangle$ fixes no point or line of Σ, so by [26] we have $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)' \rangle = G$. □

4.2 LEMMA.

(i) $\langle P_0, P_1 \rangle = G$.

(ii) *G has exactly two conjugacy classes of subgroups of index q^2+q+1, represented by $P_0\mathrm{C}_G(\tau)$ and $P_1\mathrm{C}_G(\tau)$. These two classes are interchanged by the transpose-inverse automorphism of G.*

(iii) *Suppose that $G \leq \mathrm{Aut}\,\Sigma$ where Σ is a projective plane of order q. Then Σ is Desarguesian. There are two equivalence classes of faithful actions of G on Π. In one such action, $P_0\mathrm{C}_G(\tau)$ (respectively, $P_1\mathrm{C}_G(\tau)$) is the stabilizer of a point (resp., a line) of Σ.*

(iv) *Suppose that $G \leq \mathrm{Aut}\,\Pi$ where Π is a projective plane, and let X be a point of Π. If the orbit X^G has length q^2+q+1, then its points are either collinear, form an arc, or generate a Desarguesian subplane of order q.*

Proof of (iv). It is convenient to let Σ be a Desarguesian plane of order q disjoint from Π, and to let G act on Σ as its little projective group in such a way that G_X fixes a point (i.e. rather than a line—see (iii)) of Σ. There exists a bijection θ from X^G to the point set of Σ which commutes with the action of G, viz. $X^{\theta g} = X^{g\theta}$ for all $g \in G$.

Now G acts 2-transitively on $(X^\theta)^G$ and hence on X^G. Therefore X^G forms a 2-design (see [2]) whose blocks are the members of l^G, where l is a given line of Π joining two given points of X^G. If l contains exactly two points of X^G then X^G is an arc. We may assume that l contains at least three distinct points X_1, X_2, $X_3 \in X^G$, and l is fixed by $\langle G_{X_i,X_j} : 1 \le i < j \le 3\rangle$. If X_1^θ, X_2^θ, X_3^θ form a triangle in Σ then $G_l \supseteq \langle G_{X_i^\theta,X_j^\theta} : 1 \le i < j \le 3\rangle = G$ (by [26,pp.239–240], observing that $\langle G_{X_1,X_2}, G_{X_1,X_3}\rangle = G_{X_1}$) and so all points in X^G lie on l. Otherwise X_1^θ, X_2^θ, X_3^θ all lie on some line l_1 of Σ, $G_l = G_{l_1}$, l contains exactly $q+1$ points of X^G and θ^{-1} is an imbedding of Σ in Π. □

Proof of Theorem 1.1. Suppose that $G \le \operatorname{Aut}\Pi$ where Π is a projective plane of order q^2. By [35,Satz 1], G leaves invariant a Desarguesian subplane Π_0 of order q, and G acts faithfully on Π_0. (However, the later proofs of [35,§4] contain flaws as pointed out by Lüneburg [24].) By [6,Satz 1], G acts transitively on the set of flags (X,l) of Π such that neither X nor l belongs to Π_0. By [24,Thm.2], Π is a Desarguesian or generalized Hughes plane as required. The converse also holds; the full collineation groups of the generalized Hughes planes were determined by Rosati [32], [33] (see also see [24,Cor.5,6]). □

Now suppose rather that $G \cong \mathrm{PGL}(3,q)$, where as before q is odd. The above notations still apply, with the following modifications:

$$Z_\tau = \left\{\mathrm{diag}(1,1,d) : d \in F^\times\right\},$$
$$\mathrm{C}_G(\tau) = \left\{\begin{pmatrix} a & b & 0\\ c & d & 0\\ 0 & 0 & 1\end{pmatrix} : \begin{matrix} a,b,c,d \in F,\\ ad-bc \ne 0\end{matrix}\right\}$$

where elements of G are now represented by matrices in $\mathrm{GL}(3,q)$, and $|G| = q^3(q^3-1)\times(q^2-1)$.

4.3 LEMMA. *The above Lemmas 4.1,2 remain valid with* $G \cong \mathrm{PGL}(3,q)$, *with the following amendments: 4.1(ii) becomes* $|\mathrm{C}_G(\tau)| = q(q+1)(q-1)^2$, $|Z_\tau| = q-1$; *and 4.1(v) becomes* $\mathrm{C}_G(\tau) \cong \mathrm{GL}(2,q)$.

5. The Groups PSU$(3,q)$, PGU$(3,q)$

In §5 we again restrict our attention to the case $q = p^m$ is odd, and let $F = \mathrm{GF}(q^2)$, $F^\times = F \setminus \{0\}$. For $A \in \mathrm{GL}(3,q^2)$ let A^{T} denote its transpose, and let $\overline{A}$ denote the matrix obtained by applying the field automorphism $a \mapsto \overline{a} = a^q$ to each entry of A. (We caution the reader that our matrix entries are from $F = \mathrm{GF}(q^2)$ rather than $\mathrm{GF}(q)$, for which reason certain authors have prefered the notation $\mathrm{PGU}(3,q^2)$ to that which we have followed.)

Let $G \cong \mathrm{PGU}(3,q)$, so that $|G| = q^3(q^3+1)(q^2-1)$. We shall represent the elements of G as matrices $A \in \mathrm{GL}(3,q^2)$ such that $AW\overline{A}^{\mathrm{T}} = W$, modulo the scalar matrices $\{aI : a \in F^\times,\ a\overline{a} = 1\}$, where $W \in \mathrm{GL}(3,q^2)$ is a suitably chosen hermitian matrix (i.e. $\overline{W}^{\mathrm{T}} = W$). We choose W and name certain elements of G as follows.

$$W = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \tau' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\tau = \mathrm{diag}(-1,1,-1), \qquad Z_\tau = \mathrm{Z}\big(\mathrm{C}_G(\tau)\big) = \big\{\mathrm{diag}(1,d,1) : d \in F^\times,\ d\overline{d} = 1\big\},$$

$$\mathrm{C}_G(\tau) = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} : a,b,c,d \in F,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\},$$

$$\mathrm{C}_G(\tau)' = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix} \in \mathrm{C}_G(\tau) \ :\ ad - bc = 1 \right\} \cong \mathrm{SL}(2,q),$$

$$Q = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & -\overline{a} \\ 0 & 0 & 1 \end{pmatrix} : \begin{matrix} a,\ b \in F, \\ a\overline{a} + b + \overline{b} = 0 \end{matrix} \right\} \in \mathrm{Syl}_p(G),$$

$$P = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} : b \in F,\ b + \overline{b} = 0 \right\} = \mathrm{Z}(Q) = Q',$$

$$\mathrm{N}_G(Q) = Q \rtimes K, \qquad K = \big\{\mathrm{diag}(d,1,d^{-q}) : d \in F^\times\big\}.$$

Likewise define $Z_\omega = \mathrm{Z}\big(\mathrm{C}_G(\omega)\big)$ for any involution $\omega \in G$.

Let Σ be a Desarguesian plane of order q^2 with points (resp., lines) represented by F-subspaces of $F^3 = \big\{(a,b,c) : a,\ b,\ c \in F\big\}$ of dimension 1 (resp., 2). Right-multiplication of elements of G on vectors of F^3 induces an action of G on Σ, and G commutes with the hermitian polarity δ, where for a point X of Σ represented by $(a, b, c) \in F^3 \setminus \{(0,0,0)\}$, we define X^δ to be the line

$$\big\{(x, y, z) \in F^3 : (x, y, z)W(\overline{a}, \overline{b}, \overline{c})^{\mathrm{T}} = 0\big\}.$$

Now Σ has $q^3 + 1$ **absolute** points with respect to δ (i.e. points X such that $X \in X^\delta$) and $q^2(q^2 - q + 1)$ **nonabsolute** lines (i.e. lines l such that $l^\delta \notin l$. These together form a 2-$(q^3+1, q+1, 1)$ design (see [2]) called a **hermitial unital** (see [5,p.104], [15,p.156]). Note that P consists of all (X, X^δ)-elations of Σ in G, and Z_τ consists of all (Y, Y^δ)-homologies of Σ in G, where $X = (0, 0, 1)$ is absolute and $Y = (0, 1, 0)$ is nonabsolute.

We state below a few facts concerning G, omitting the proofs of those statements which are well known or which follow readily from [16], [26,p.241], [29].

5.1 LEMMA.

(i) *G has a single conjugacy class of involutions, and G acts transitively by conjugation on the set of ordered pairs of commuting distinct involutions. One such pair is $\{\tau, \tau'\}$ where $\tau' = \mathrm{diag}(-1, 1, -1)$.*

(ii) $|\mathrm{C}_G(\tau)| = q(q+1)^2(q-1), \quad |Z_\tau| = q + 1.$

(iii) $\mathrm{C}_G(\tau) = \mathrm{C}_G(\tau)' \rtimes Z_{\tau'}$.

(iv) $\mathrm{C}_G(\tau)/Z_\tau \cong \mathrm{PGL}(2, q)$.

(v) *$\mathrm{C}_G(\tau)$ is the unique maximal subgroup of G containing $\mathrm{C}_G(\tau)'$.*

(vi) *G contains involutions τ_1, τ_2 such that $\mathrm{C}_G(\tau_1)' \cap \mathrm{C}_G(\tau_2)' \neq 1$, $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)' \rangle = G$.*

(vii) *If $e > 2$ is an even divisor of $q + 1$ then there exists an involution $\tau'' \in \mathrm{C}_G(\tau)$ and elements $x \in Z_{\tau'}$, $y \in Z_{\tau''}$ of order e such that $\langle x, y \rangle \supseteq \mathrm{C}_G(\tau)'$.*

(viii) *Suppose that $q = 3$ and $\omega \in G$ is an involution. Then $\mathrm{O}_2\big(\mathrm{C}_G(\omega)'\big) = \mathrm{O}_2\big(\mathrm{C}_G(\omega)\big)$ is quaternion. Furthermore if $[\tau, \omega] = 1$ then $\big\langle \mathrm{O}_2\big(\mathrm{C}_G(\tau)\big), \mathrm{O}_2\big(\mathrm{C}_G(\omega)\big)\big\rangle = \mathrm{N}_G\big(\langle \tau, \omega \rangle\big)$ of order 96; if $[\tau, \omega] \neq 1$ then $\big\langle \mathrm{O}_2\big(\mathrm{C}_G(\tau)\big), \mathrm{O}_2\big(\mathrm{C}_G(\omega)\big)\big\rangle = G$.*

(ix) *If $q = 3$ then there exists an involution $\tau'' \in \mathrm{C}_G(\tau)$ such that $[\tau', \tau''] \neq 1$, $\tau'\tau'' \in \mathrm{O}_2\big(\mathrm{C}_G(\tau)\big)$.*

Proof of (vi), (vii), (ix). Let l_0 be an absolute line of Σ, and let $X_0 = l_0^\delta$, X_1, X_2 be three distinct points of l_0. Then (vi) follows as in 4.1(vi).

By 3.2 there exist x, $y \in \mathrm{C}_G(\tau)$ such that $\overline{x}$, $\overline{y} \in \overline{\mathrm{C}_G(\tau)}$ are of order e, and $\langle \overline{x}, \overline{y} \rangle \supseteq \overline{\mathrm{C}_G(\tau)'} \cong \mathrm{PSL}(2, q)$ where the bars indicate the canonical images in $\mathrm{C}_G(\tau)/Z_\tau \cong \mathrm{PGL}(2, q)$. Now $Z_{\tau'}$ contains an element u of order e, and since $Z_{\tau'} \cap Z_\tau = 1$, the image $\overline{u} \in \overline{\mathrm{C}_G(\tau)}$ is also of order e. Since $\overline{\mathrm{C}_G(\tau)}$ has a single conjugacy class of cyclic subgroups of order e, we may assume that $x \in Z_{\tau'}$, and we also have $y \in Z_{\tau''}$ for some involution $\tau'' \in \mathrm{C}_G(\tau)$, and x, y have order e.

Now $\langle x, y \rangle Z_\tau \supseteq \mathrm{C}_G(\tau)' Z_\tau$ which yields $\langle x, y \rangle \supseteq \big(\mathrm{C}_G(\tau)'\big)'$. If $q > 3$ then $\big(\mathrm{C}_G(\tau)'\big)' = \mathrm{C}_G(\tau)'$ and so we are done. If $q = 3$ then $\langle x, y \rangle \supseteq \big(\mathrm{C}_G(\tau)'\big)'$, the latter being quaternion; but also $\langle x, y \rangle$ contains an element of order 3, so that $\langle x, y \rangle \supseteq \mathrm{C}_G(\tau)'$ and in any case (vii) holds.

If $q = 3$ then

$$\mathrm{O}_2\big(\mathrm{C}_G(\tau)\big) = \left\langle \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ i & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & -i \\ 0 & 1 & 0 \\ i & 0 & 1 \end{pmatrix} \right\rangle$$

where $i \in F$, $i^2 = -1$, and so (ix) follows by taking

$$\tau'' = \begin{pmatrix} i & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & -i \end{pmatrix}.$$

□

5.2 LEMMA.

(i) *K is cyclic of order $q^2 - 1$. For $d \mid q^2 - 1$ let K_d be the subgroup of order d in K. Then $Z_\tau = K_{q+1}$.*

(ii) *K acts irreducibly on the vector space Q/P of dimension $2m$ over* GF*(p).*

(iii) $\mathrm{C}_G(\tau) = \mathrm{C}_G(\tau)' K$.

5.3 LEMMA.

(i) *G has a single conjugacy class of subgroups of index $q^3 + 1$, represented by* $\mathrm{N}_G(Q)$.

(ii) *Suppose that $G \leq \mathrm{Aut}\,\Pi$ where Π is a projective plane, and let X be a point of Π. If the orbit X^G has length $q^3 + 1$, then its points are either collinear, form an arc, or form the point set of a hermitian unital embedded in Π.*

Proof of (ii). There exists a bijection θ from X^G to the set of absolute points of Σ with respect to δ, such that θ commutes with the action of G, i.e. $X^{\theta g} = X^{g\theta}$ for all $g \in G$. The result follows as in the proof of 4.2(iv), using instead p.241 of [26]. □

5.4 LEMMA. *The above Lemmas 5.1–3 remain valid with $G \cong \mathrm{PSU}(3, q)$, with the following amendments: 5.1(ii) becomes $|\mathrm{C}_G(\tau)| = q(q + 1)^2(q - 1)/\mu$, $|Z_\tau| = (q + 1)/\mu$ where $\mu = (q+1, 3)$; 5.1(vii) requires the additional hypothesis that $e \mid (q+1)/\mu$ (and in particular $q \neq 5$); $|K| = (q^2 - 1)/\mu$ and $Z_\tau = K_{(q+1)/\mu}$ in 5.2(i).*

6. Abelian Planar Collineation Groups

Recall that a collineation group G of a projective plane Π is **planar** if $\mathrm{Fix}(G)$ is a subplane of Π. That such collineation groups must often be considered is evident from 2.1. Our results concern the simplest such case, in which G is abelian. For example the following is proven in [28].

6.1 THEOREM. *If G is a faithful abelian planar collineation group of a projective plane Π of order n, then $|G| < n$.*

Suppose now that G is a faithful abelian planar collineation group of a finite projective plane Π, and let $\Pi_G = \mathrm{Fix}(G)$. If Π_1, Π_2 are subplanes of Π, then we shall denote by $\langle \Pi_1, \Pi_2 \rangle$ the subplane generated by Π_1 and Π_2; we shall write $\Pi_1 \subseteq \Pi_2$ if Π_1 is a subplane of Π_2. For any subplane $\Sigma \subseteq \Pi$, let

$$G_\Sigma = \{g \in G : g \text{ fixes } \Sigma \text{ pointwise}\}, \qquad \mathcal{G} = \{G_\Sigma : \Sigma \subseteq \Pi\}.$$

For any subgroup $H \le G$, let

$$\Pi_H = \mathrm{Fix}(H), \qquad \mathcal{P} = \{\Pi_H : H \le G\};$$

note that $\mathcal{P}$ consists of certain subplanes of Π containing Π_G. We consider $\mathcal{G}$, $\mathcal{P}$ as **posets** (i.e. partially ordered sets, with respect to inclusion denoted as usual by $\subseteq$). Let $\mathrm{St} : \mathcal{P} \to \mathcal{G}$ (abbreviation for 'stabilizer') denote the restriction to $\mathcal{P}$ of the map $\Sigma \mapsto G_\Sigma$, and let the restriction $\mathrm{Fix}\big|_{\mathcal{G}}$ also be denoted by Fix, so that $\mathrm{Fix} : \mathcal{G} \to \mathcal{P}$ is the map $H \mapsto \Pi_H$. The following properties may be immediately verified.

(i) $\mathcal{G}$ contains both $G = \mathrm{St}(\Pi_G)$ and $1 = \mathrm{St}(\Pi)$;
$\mathcal{P}$ contains both $\Pi = \mathrm{Fix}(1)$ and $\Pi_G = \mathrm{Fix}(G)$.

(ii) G leaves invariant every member of $\mathcal{P}$ (because G is abelian).

(iii) Fix reverses inclusion, i.e. $\Pi_H \subseteq \Pi_K$ whenever $H \supseteq K$, $H, K \in \mathcal{G}$.

(iv) St reverses inclusion, i.e. $G_\Sigma \subseteq G_{\Sigma'}$ whenever $\Sigma \supseteq \Sigma'$, Σ, $\Sigma' \in \mathcal{P}$.

(v) $\mathrm{Fix} \circ \mathrm{St} = \mathrm{id}_{\mathcal{P}}$, $\mathrm{St} \circ \mathrm{Fix} = \mathrm{id}_{\mathcal{G}}$, so that $\mathrm{St} : \mathcal{P} \to \mathcal{G}$ and $\mathrm{Fix} : \mathcal{G} \to \mathcal{P}$ are anti-isomorphisms of posets.

(vi) $G_{\langle \Sigma, \Sigma' \rangle} = G_\Sigma \cap G_{\Sigma'}$ whenever Σ, $\Sigma' \in \mathcal{P}$; thus $\mathcal{G}$ is closed under intersection.

(vii) $\Pi_{H \cap K} = \langle \Pi_H, \Pi_K \rangle$ whenever H, $K \in \mathcal{G}$; thus $\langle \Sigma, \Sigma' \rangle \in \mathcal{P}$ whenever Σ, $\Sigma' \in \mathcal{P}$.

We caution the reader that $\mathcal{G}$, $\mathcal{P}$ need not be lattices: namely if Σ, $\Sigma' \in \mathcal{P}$ then $\Sigma \cap \Sigma' \subseteq \mathrm{Fix}\langle G_\Sigma, G_{\Sigma'} \rangle$, and if the latter inclusion is proper then $\mathcal{P}$ is not closed under the usual intersection. Especially note that our $\mathcal{P}$ is *not* the lattice of all G-invariant substructures of Π (or even a sublattice thereof) as considered in [13,§4].

Applying Theorem 6.1 to the action of G/H on Π_H for $H \in \mathcal{G}$, we obtain

(viii) $[G\!:\!H] < n_H$ for all $H \in \mathcal{G}$, where n_H is the order of Π_H.

If H, $K \in \mathcal{G}$ (and similarly for members of $\mathcal{P}$) we shall write $H \prec K$ in case $H \subsetneq K$ and there is no $L \in \mathcal{G}$ satisfying $H \subsetneq L \subsetneq K$. Whenever H, $K \in \mathcal{G}$ we clearly have

(ix) $H \prec K$ if and only if $\Pi_K \prec \Pi_H$.

Suppose that H, $K \in \mathcal{G}$, $H \prec K$ and let Π_H, Π_K have order n_H, n_K respectively. Let l be a line of Π_G, so that l belongs to Π_H, Π_K. If X is a point of l in Π_H outside Π_K, then the orbit X^K consists of $[K\!:\!K_X]$ points of l, all of which are fixed by K_X since G is abelian. Thus K_X fixes pointwise the subplane $\langle \Pi_K, X^K \rangle \subseteq \Pi_H$, and since $\Pi_K \prec \Pi_H$ we obtain $\langle \Pi_K, X^K \rangle = \Pi_H$. This yields $K_X = H$, and since every K-orbit on the points of l in Π_H but outside Π_K has length $[K\!:\!H]$, we conclude that

(x) $[K\!:\!H] \,\big|\, n_H - n_K$ whenever $H \prec K$, where Π_H, Π_K has order n_H, n_K respectively.

Choose a maximal chain in $\mathcal{P}$, namely

$$\Pi_G = \Pi_0 \prec \Pi_1 \prec \cdots \prec \Pi_k = \Pi, \qquad \Pi_i \in \mathcal{P},\ i = 0, 1, \ldots, k,$$

and let n_i be the order of Π_i, $i = 0, 1, \ldots, k$. Then $n_{i-1}^2 \le n_i$, $i = 1, 2, \ldots, k$ by 2.2, so that by induction we obtain

(xi) the length k of any chain in $\mathcal{P}$ (or in $\mathcal{G}$) satisfies $n_G^{2^k} \le n$, where Π, Π_G has order n, n_G respectively.

We make use of the above concepts in proving the following.

6.2 THEOREM. *Suppose that P is an elementary abelian group of order $q = p^m$, p a prime, and that $P \trianglelefteq G$ where the group G acts transitively by conjugation on the cyclic subgroups of P. Suppose furthermore that $G \le \mathrm{Aut}\,\Pi$ for some projective plane Π of order q^2, such that P fixes pointwise a subplane Π_P of order n_P. Then one of the following must hold:*

(I) Π_P *is a Baer subplane of* Π, *or*

(II) q *is a square,* $n_P = \sqrt{q}$, *and* P *has a subgroup of order* $\sqrt{q}$ *fixing pointwise a Baer subplane of* Π.

We illustrate 6.2 by listing some known occurrences for $q \le 4$. If $q = 2$ and $G = P \cong \mathrm{C}_2$, then case (I) occurs for the unique (Desarguesian) plane of order 4. If $q = 3$ and $G = P \cong \mathrm{C}_3$ then case (I) occurs for the Hughes plane of order 9 (see [24,Cor.5]); also for the Hall and dual Hall plane of order 9.

For $q = 4$ the following translation planes (or their duals) of order 16 (see [8]) admit $G \cong \mathrm{A}_4$ as in Theorem 6.2. If Π is a Hall plane, a derived semifield plane or a Dempwolff plane then case (I) occurs; if Π is a Lorimer-Rahilly plane or a Johnson-Walker plane then case (I) or case (II) may occur.

Proof of Theorem 6.2. For $g \in P \setminus 1$, let n_1 be the order of the subplane $\Pi_g = \mathrm{Fix}(g)$. (By the action of G on P, n_1 is independent of the choice of $g \in P \setminus 1$.) Let l be a line of Π_P. Counting in two different ways the number of pairs (X, g) such that X is a point of l, $g \in P$ and $X^g = X$, we obtain

$$q^2 + 1 + (q-1)(n_1 + 1) = w|P|$$

where w is the number of orbits of P on the points of l (see [30]). This gives $q \mid n_1$, and since $n_1 \le q$, we have $n_1 = q$.

Let P_g be the kernel of the action of P on Π_g, so that

$$P_g \in \mathcal{G} = \{P_\Sigma : \Sigma \subseteq \Pi\}, \qquad P_\Sigma = \{h \in P : h \text{ fixes } \Sigma \text{ pointwise}\}.$$

(We follow the notation used under 6.1, except that our abelian planar collineation group is now P in place of G.) Note that $\mathcal{G}$ is invariant under the action of G by conjugation on the subgroups of P.

Clearly $P_g \succ 1$, and so for any $H \in \mathcal{G}$ we have $H \cap P_g =$ either 1 or P_g. This means that any $H \in \mathcal{G}$ is a disjoint union of certain conjugates of P_g in G. Writing $|P_g| = p^r$, $|H| = p^s$, this means that $p^r - 1 \mid p^s - 1$, i.e. $r \mid s$ so that $|H| = u^d$ for some integer $d \geq 0$ where $u = |P_g| = p^r$. In particular $|P| = u^e$ for some integer $e \geq 1$.

If $P_g = P$ we have case (I); hence we may assume that $P_g \subsetneq P$, $e \geq 2$. If $P_g \prec P$ then by (x) we have $u^{e-1} = [P:P_g] \mid u^e - n_P$, $u^{e-1} \mid n_P$; but $n_P \leq \sqrt{n_1} = \sqrt{q} = u^{e/2}$ so that $e = 2$, $n_P = \sqrt{q}$ and we have case (II).

Hence we may assume that $1 \prec P_g \prec H \subsetneq P$ for some $H \in \mathcal{G}$, $|H| = u^d$. By (viii) we have $[P:H] < n_H \leq \sqrt{n_1} = \sqrt{q}$ where n_H is the order of Π_H, i.e. $u^d = |H| > \sqrt{q} = u^{e/2}$, $2d > e$. Choose $x \in G$ such that $H^x \neq H$; then

$$H^x \cap H \in \mathcal{G}, \qquad |H^x \cap H| = \frac{|H|^2}{|H^x H|} \geq u^{2d-e} \geq u.$$

We may assume that $g \in H^x \cap H$; otherwise replace g by g^y where $y \in G$ is chosen such that $g^y \in H^x \cap H$. Now $P_g \subseteq H^x \cap H \subsetneq H$ and so $H^x \cap H = P_g$ which forces $2d - e = 1$, $d = \frac{1}{2}(e+1)$ and in particular e is odd, $e \geq 3$.

Suppose that $H \subsetneq K \subsetneq P$ for some $K \in \mathcal{G}$. Choose $z \in G$ such that $K^z \not\supseteq H$; then

$$K^z \cap H \in \mathcal{G}, \qquad |K^z \cap H| = \frac{|K^z||H|}{|K^z H|} > \frac{|H|^2}{|K^z H|} \geq u^{2d-e} = u.$$

Again we may assume that $g \in K^z \cap H$; then $P_g \subsetneq K^z \cap H \subsetneq H$, a contradiction.

Therefore $1 \prec P_g \prec H \prec P$, and so (x) gives $u^{(e-1)/2} = [P:H] \mid n_H - n_P$, $u^{(e-1)/2} = [H:P_g] \mid u^e - n_H$ so that $u^{(e-1)/2} \mid n_P$. By (xi) we have $\left(u^{(e-1)/2}\right)^8 \leq n_P^8 \leq u^{2e}$, $e \leq 2$, a final contradiction. □

7. Proof of Theorem 1.8

The result is easily established for $q = 3$ (see Prop. 2.7 of [28]) so we may assume that

(1) $q > 3$.

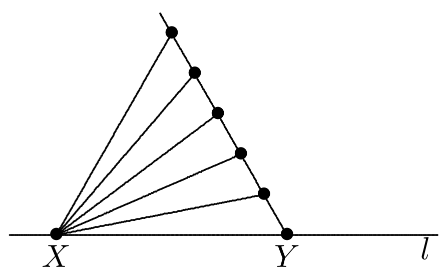
We next show that

(2) G' acts irreducibly on Π.

Since G' induces $G'/G' \cap K \cong \mathrm{PSL}(2, q)$ on Π, (2) follows by Theorem 1.7 for $q \neq 5, 9$.

Suppose that $q = 5$ and that G' fixes a line l of Π. By Theorem 1.7, G' has orbits of length 5, 5, 6, 10 on the points of l, and G permutes these orbits. Indeed, G has the same four orbits on l, since the G'-orbits of length 5 may be represented by X, Y respectively, where $\mathrm{Fix}(\tau, \tau')$ is given by Figure 7A for commuting involutions $\tau \neq \tau'$ in G', and from the lack of symmetry in X and Y it is apparent that G preserves both $X^{G'}$ and $Y^{G'}$.

FIGURE 7A. Exceptional $\mathrm{Fix}(\tau, \tau')$ for $q = 5$



We may write $G = G' \rtimes \langle g \rangle$ where g is of order 4 and induces an automorphism of order 4 on $G'/G' \cap K$ (cf. 4.1(iii)). This automorphism leaves invariant exactly one subgroup of isomorphism type A_4, two dihedral subgroups of order 10, and none of type S_3 in $G'/G' \cap K \cong \mathrm{A}_5$. Thus g induces a collineation of Π of order 4 fixing exactly 4 points of l, which is clearly impossible.

Now suppose that $q = 9$. Then $|\mathrm{Z}(G)| = 8$, and if $K \subsetneq \mathrm{Z}(G)$ then $\mathrm{Z}(G)$ contains an element g inducing an involutory collineation of Π. If g induces a homology of Π then G' fixes its centre and axis, contrary to Theorem 1.7. Otherwise $\mathrm{Fix}(g)$ is a subplane of order 9 on which G' acts reducibly, contrary to 1.6.

Therefore $K = \mathrm{Z}(G)$, i.e. G induces $\overline{G} = G/\mathrm{Z}(G) \cong \mathrm{PGL}(2, 9)$ on Π. By (16) of [28], the lengths of the orbits of $\overline{G'} \cong \mathrm{PSL}(2, 9)$ on the points of l are given by one of the following cases:

lengths 1, 36,45 (in case (ix) of [28,(16)]);

lengths 1, 15, 30, 36 (in cases (x), (xi));

lengths 1, 6, 15, 60 (in cases (xii), (xiii)); or

lengths 1, 6, 15, 20, 40 (in cases (xiv), (xv)).

If case (x) or (xi) occurs then the unique $\overline{G'}$-orbit of length 30 on l is $\overline{G}$-invariant. From Table 3D of [28] we see that ρ_1, ρ_2 fix 6, 0 points in this orbit, respectively, violating the fact that ρ_1, ρ_2 are conjugate in $\overline{G}$. We similarly eliminate cases (xii)–(xv) of [28,(16)].

We are left with case (ix), and $\overline{G'}$ has three orbits on the points of l, of length 1, 45, 36 respectively, and so each of these three orbits is $\overline{G}$-invariant. The stabilizers in $\overline{G}$ of point representatives from these orbits are $\overline{G}$, dihedral of order 16, and dihedral of order 20 respectively. We compute (cf. (9) of [28]) that an involution $\omega \in \overline{G} \setminus \overline{G'}$ fixes 1, 5, 6 points in these orbits respectively, so that ω fixes exactly 12 points of l, which is clearly impossible. This concludes the proof of (2).

Let

$$
\begin{aligned}
P &= \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathrm{GF}(q) \right\} \in \mathrm{Syl}_p(G), \qquad P < G', \\
Z &= \left\{ \mathrm{diag}(d,d) : d \in \mathrm{GF}(q) \setminus \{0\} \right\} = \mathrm{Z}(G), \\
C &= \left\{ \mathrm{diag}(d,1) : d \in \mathrm{GF}(q) \setminus \{0\} \right\}, \\
N &= \mathrm{N}_G(P) = CZP, \\
\gamma &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in C, \qquad \tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

(3) We may assume that $\mathrm{Fix}(N) = \emptyset$.

For otherwise, by duality we may suppose that N fixes a point X of Π. Since N is a maximal subgroup of G, (2) gives $G_X = N$. By (2), not all $q+1$ points of X^G are collinear, and so [28,Prop.2.9] gives conclusion (i) of 1.8 and we are done. This proves assertion (3).

Since every P-orbit has length either 1 or a multiple of p, $\mathrm{Fix}(P)$ is neither empty nor a triangle. Hence by (3) and 2.1 we conclude that $\mathrm{Fix}(P)$ is a subplane of Π. Since C acts transitively on $P \setminus 1$ by conjugation, Theorem 6.2 yields

(4) Fix(P) is a subplane of Π of order n_P, where $n_P \in \{\sqrt{q},\, q\}$.

Suppose that X is a point of Π with X^G an arc, $|X^G| > q+1$. If $g \in P \setminus 1$ then Fix(g) is a subplane of order q by the proof of Theorem 6.2. There certainly exists a point of X^G outside Fix(g); thus $X^h \notin$ Fix(g) for some $h \in G$. Since X^h lies on a unique line of Fix(g), this line contains at least $|\langle g\rangle| > 2$ points of X^G. Thus

(5) for any point X of Π such that $|X^G| > q+1$, X^G is not an arc.

Clearly

(6) the permutation group induced by N on the points of Fix(P) is abelian of order dividing $\frac{1}{2}(q-1)^2$;

namely, the induced permutation group is a homomorphic image of $N/KP \cong CZ/K$.

(7) One of the following must occur:

(I) $n_P = q$, γ induces a Baer collineation of Fix(P) (i.e. Fix(P,γ) is a subplane of order $\sqrt{q}$);

(II) $n_P = \sqrt{q}$, γ acts trivially on Fix(P) (i.e. Fix$(\gamma) \supsetneq$ Fix(P)); or

(III) $n_P = \sqrt{q}$, γ induces a Baer collineation of Fix(P) (i.e. Fix(P,γ) is a subplane of order $q^{1/4}$).

To see this, note firstly that γ cannot induce a homology on Fix(P) (for otherwise its centre would be fixed by N, contrary to (3)). Secondly if $n_P = q$, then γ cannot act trivially on Fix(P) (or else Fix$(\gamma) =$ Fix(P), but then since $\gamma^\tau \equiv \gamma \bmod K$ we obtain Fix$(P^\tau) =$ Fix$(\gamma^\tau) =$ Fix$(\gamma) =$ Fix(P), i.e. a Baer subplane of Π is fixed pointwise by $\langle P^\tau, P\rangle = G'$, contradicting (2)). This gives (7), and as a corollary,

(8) q is a square; in particular $q \equiv 1 \bmod 8$.

Next we show that

(9) in case (7,II) we have $q \notin \{9,\ 25,\ 121\}$.

For suppose that case (7,II) occurs with $q = 121$. Now Fix(P) is of order 11, and by (6) the group induced by N on Fix(P) is abelian of order dividing $2^5{\cdot}3^2{\cdot}5^2$. If $g \in N$ induces an involutory collineation of Fix(P), then g induces a homology of Fix(P) whose centre is fixed by N, contrary to (3).

Suppose that $g \in N$ induces a collineation of order 5 on Fix(P). Since $N = CZP$, we may assume that $g \in CZ = \mathrm{N}_G(P) \cap \mathrm{N}_G(P^\tau)$. By (3), Fix$(P, g)$ must be a triangle with vertices X_0, X_1, X_2, say. Since Fix(P), Fix(P^τ) are disjoint Baer subplanes of Fix(γ), there is a unique point Y_j of Fix(P^τ) on the line X_jX_{j+1}, $j = 0,\ 1,\ 2$ (subscripts modulo 3). Now g leaves Fix(P^τ) invariant and so fixes Y_0, Y_1, Y_2. But g induces a triangular collineation of Fix(P^τ) (for otherwise g acts trivially on the subplane generated by Fix$(P^\tau) \cup \{X_0, X_1, X_2\}$, i.e. on Fix$(\gamma)$, contradicting the assumption that g acts nontrivially on Fix(P)). Also, g^τ induces a triangular collineation of Fix(P^τ); namely, Fix(P^τ, g^τ) is the triangle $X_0^\tau X_1^\tau X_2^\tau$. Since the actions of g, g^τ on Fix(P^τ) commute, we must have $\{X_0^\tau,\ X_1^\tau,\ X_2^\tau\} = \{Y_0,\ Y_1,\ Y_2\}$. But this means that the triangle $X_0^\tau X_1^\tau X_2^\tau$ is inscribed in the distinct triangle $X_0X_1X_2$, and by applying τ we see that the triangle $X_0X_1X_2$ is likewise inscribed in $X_0^\tau X_1^\tau X_2^\tau$, which is absurd.

Hence the group induced by N on Fix(P) has order dividing 9, and so N fixes at least one of the 133 points of Fix(P), contradicting (3).

The cases $q = 9$, 25 are eliminated with much less difficulty, as the reader may verify, and in any case (9) holds.

(10) N has no orbit of length 3 on the points (or lines) of Fix(P).

For suppose that $\{X_0,\ X_1,\ X_2\}$ are three points of Fix(P) which form an orbit under N. By (3) these three ponts are not collinear, and hence form a triangle. By (6) these three points have the same stabilizer N_0 in N. Now $N_0 \supseteq KP$, $[N : N_0] = 3$ and (6) gives

$p \neq 3$. Since N is the unique maximal subgroup of G containing N_0, we have $N_0 = G_{X_0}$, $|X_0^G| = 3(q{+}1)$. Let $y \in N \setminus N_0$, and by proper choice of subscripts, we may assume that $X_j^y = X_{j+1}$, $j = 0, 1, 2$ (subscripts modulo 3). Since $N = CZP$, we may assume that $y \in CZ = \mathrm{N}_G(P) \cap \mathrm{N}_G(P^\tau)$. Since $\tau \in \mathrm{N}_G(\langle y \rangle)$, $\{X_0^\tau, X_1^\tau, X_2^\tau\}$ is also a $\langle y \rangle$-orbit forming the vertices of a triangle. We claim that $\{X_0, X_1, X_2, X_0^\tau, X_1^\tau, X_2^\tau\}$ is an arc. If not then by symmetry, we may suppose that some point of $\{X_0^\tau, X_1^\tau, X_2^\tau\}$ lies on the line X_0X_1, and the action of y shows that the triangle $X_0^\tau X_1^\tau X_2^\tau$ is inscribed in the triangle $X_0X_1X_2$. But then an application of τ shows that $X_0X_1X_2$ is likewise inscribed in $X_0^\tau X_1^\tau X_2^\tau$, which is absurd. Hence $\{X_0, X_1, X_2, X_0^\tau, X_1^\tau, X_2^\tau\}$ is a 6-arc as claimed.

By (5) we may choose three distinct collinear points $Y_0, Y_1, Y_2 \in X_0^G$. Let P_j be the (unique) Sylow p-subgroup of G fixing Y_j, $j = 0, 1, 2$. The previous paragraph shows that we *cannot* have $P_0 = P_1 \neq P_2$. Of course we cannot have $P_0 = P_1 = P_2$ (since the three points of X_0^G belonging to $\mathrm{Fix}(P_0)$ form a triangle). Hence P_0, P_1, P_2 are distinct. We may assume that $P_0 = P$, $Y_0 = X_0$, $P_1 = P^\tau$. For every $g \in N$ whose order is not divisible by 3, (6) yields $g \in N_0$. Choosing involutions $\tau_j \in \big(\mathrm{N}_{G'}(P_j) \cap \mathrm{N}_{G'}(P_{j+1})\big) \setminus Z$, $j = 0, 1, 2$ (subscripts modulo 3), this means that $Y_j^{\tau_j} = Y_j$, $Y_{j+1}^{\tau_j} = Y_{j+1}$. (Note that (8) guarantees the existence of such involutions τ_j.) Therefore the line l joining Y_0, Y_1, Y_2 is fixed by $\tau_j\tau_{j+1} \in P_{j+1} \setminus 1$, $j = 0, 1, 2$ (note that τ_j, τ_{j+1} both invert each element of P_{j+1}, so that $\tau_j\tau_{j+1} \in \mathrm{C}_{G'}(P_{j+1}) = P_{j+1}$). We have the stabilizers $(P_0)_l \neq 1$, $(P_1)_l \neq 1$; however $(P_0)_l \subsetneq P_0$ and $(P_1)_l \subsetneq P_1$, for otherwise l is fixed by G', contradicting (2). Hence either case (I) or (II) of (7) occurs, and $|(P_0)_l| = |(P_1)_l| = \sqrt{q}$. Writing $S = \langle (P_0)_l, (P_1)_l \rangle \subseteq G_l$, we have $S \subsetneq G' \cong \mathrm{SL}(2, q)$ and so the classification of subgroups of $\mathrm{SL}(2, q)$ (see [34]) gives $S \cong \mathrm{SL}(2, \sqrt{q})$. The stabilizer $(P_0)_{Y_1} = 1$; for otherwise Y_1 is fixed by $\langle P_1, (P_0)_{Y_1} \rangle = G'$, contrary to (2). Hence l contains at least $\sqrt{q} + 1$ members of X_0^G, namely $\{Y_0\} \cup \{Y_1^g : g \in (P_0)_l\}$. On the other hand, if $Y_3 \in X_0^G$ lies on l, and P_3 is the unique Sylow p-subgroup of G fixing Y_3, then the previous argument shows that $|(P_3)_l| = \sqrt{q}$ and $(P_3)_l \in \mathrm{Syl}_p(S)$. Since S has only $\sqrt{q} + 1$ Sylow p-subgroups, this means that l carries exactly $\sqrt{q} + 1$ points of X_0^G.

Now consider the lines $l_j = X_0X_j^\tau$, $j = 0, 1, 2$. Let k_j be the number of points of X_0^G on l_j, and let r_j be the number of lines of l_j^G through X_0. We have seen that

$k_j \in \{2, \sqrt{q}+1\}$ for $j = 0, 1, 2$ and that at least one of k_0, k_1, k_2 equals $\sqrt{q}+1$. There are three cases to consider:

(a) $l_0^G = l_1^G = l_2^G$;

(b) l_0^G, l_1^G, l_2^G are mutually distinct; or

(c) two of l_0^G, l_1^G, l_2^G coincide and the third is distinct.

We shall examine each of these cases in turn, by counting in two different ways each of the quantities

$$n_1 = \bigl|\bigl\{(X,l) \in X_0^G \times l_j^G : X \in l_j\bigr\}\bigr|,$$
$$n_2 = \bigl|\bigl\{(X,Y) \in X_0^G \times X_0^G : X \neq Y,\ XY \in l_j^G\bigr\}\bigr|$$

where we now fix a subscript j such that $k_j = \sqrt{q}+1$. In case (a), $k_0 = k_1 = k_2 = \sqrt{q}+1$ and we obtain

$$n_1 = 3(q+1)r_0 = |l_0^G|\, k_0,$$
$$n_2 = 3(q+1)\cdot 3q = |l_0^G|\, k_0(k_0-1);$$

hence $r_0 = 3\sqrt{q}$ and our expression for n_1 yields $\sqrt{q}+1 \bigm| 9\sqrt{q}(q+1)$ so that $\sqrt{q}+1 \bigm| 18$ which yields $q = 25$, contrary to (9).

In case (b) we obtain

$$n_1 = 3(q+1)r_j = |l_j^G|\, k_j,$$
$$n_2 = 3(q+1)q = |l_j^G|\, k_j(k_j-1);$$

hence $r_j = \sqrt{q}$, $\sqrt{q}+1 \bigm| 3\sqrt{q}(q+1)$ which leads to a contradiction as in (a).

In case (c) we may assume that l_j^G coincides with precisely one of l_{j+1}^G, l_{j+2}^G (subscripts modulo 3); for otherwise $l_j^G \neq l_{j+1}^G = l_{j+2}^G$ and so n_1, n_2 are precisely as in (b), a contradiction. Therefore we have

$$n_1 = 3(q+1)r_j = |l_j^G|\, k_j,$$
$$n_2 = 3(q+1)\cdot 2q = |l_j^G|\, k_j(k_j-1);$$

hence $r_j = 2\sqrt{q}$, $\sqrt{q}+1 \bigm| 6\sqrt{q}(q+1)$ so that $\sqrt{q}+1 \bigm| 12$, $q \in \{9, 25, 121\}$, again contradicting (9). This completes the proof of (10).

By (7), N acts on a subplane Π_1 of order $\sqrt{q}$: in case (7, I) we let $\Pi_1 = \mathrm{Fix}(P,\gamma)$; in cases (II), (III) of (7) let $\Pi_1 = \mathrm{Fix}(P)$. By (3), (6), (10) we conclude that no subgroup of N fixes precisely a triangle of Π_1. By 2.1 this means that

(11) for any $H \le N$, $\mathrm{Fix}_{\Pi_1}(H)$ is either empty or a (not necessarily proper) subplane of Π_1.

(Here $\mathrm{Fix}_{\Pi_1}(H)$ denotes $\Pi_1 \cap \mathrm{Fix}(H)$.) Letting D_r be the Sylow r-subgroup of CZ for each prime $r \,\big|\, q-1$, we have

(12) D_r fixes pointwise a (not necessarily proper) subplane of Π_1 for $r \ne 3$. The order k_r of this subplane satisfies $r \,\big|\, k_r \pm 1$ according as $r \,\big|\, \sqrt{q} \pm 1$.

If $\mathrm{Fix}_{\Pi_1}(D_r) = \varnothing$, then $|D_r| \,\big|\, (q-1, q+\sqrt{q}+1)$, i.e. $|D_r| \in \{1, 3\}$, contrary to assumption. Hence by (11), $\mathrm{Fix}_{\Pi_1}(D_r)$ is a subplane of Π_1. If k_r is its order, we clearly have $r \,\big|\, \sqrt{q} - k_r$ from which (12) follows.

We may factorise $N = CZP$, $CZ = D_0 D_1 D_3$ where D_1 is the product of the Sylow r-subgroups of CZ as r ranges over all primes $r \,\big|\, q-1$ such that $r \equiv 1 \bmod 3$, $r^2 - r + 2 \le \sqrt{q}$; and $D_0 \cap D_1 D_3 = 1$. We claim that

(13) $\mathrm{Fix}_{\Pi_1}(D_1) = \varnothing$.

If $\mathrm{Fix}_{\Pi_1}(D_0 D_1) \ne \varnothing$ then (11) implies that $\mathrm{Fix}_{\Pi_1}(D_0 D_1)$ is a subplane of Π_1, of order k, say. By (3) every point orbit of D_3 on $\mathrm{Fix}_{\Pi_1}(D_0 D_1)$ has length 3^e for some $e \ge 1$. But $9 \nmid k^2 + k + 1$, so D_3 has at least one point orbit of length 3 on $\mathrm{Fix}_{\Pi_1}(D_0 D_1)$, contrary to (10). Therefore $\mathrm{Fix}_{\Pi_1}(D_0 D_1) = \varnothing$.

We complete the proof of (13) by induction on the number of prime divisors of $|D_0|$. Accordingly, suppose that $\mathrm{Fix}_{\Pi_1}(D_r D^*) = \varnothing$ where $D^* \le CZ$ and r is some prime divisor of $q-1$ such that $r \equiv 2 \bmod 3$, or $r \equiv 1 \bmod 3$ and $r^2 - r + 2 > \sqrt{q}$. We must show that $\mathrm{Fix}_{\Pi_1}(D^*) = \varnothing$. If not, then (11) implies that $\mathrm{Fix}_{\Pi_1}(D^*)$ is a subplane of Π_1, of order k, say. Now D_r acts on $\mathrm{Fix}_{\Pi_1}(D^*)$ without fixing any point, so that $r \mid k^2 + k + 1$. Since $r \neq 3$ this means that GF(r) contains a nontrivial cube root of 1. Hence $r \equiv 1 \bmod 3$ and $r^2 - 2 + 2 > \sqrt{q}$. Also since D_r acts nontrivially on Π_1, (12) implies that $\mathrm{Fix}_{\Pi_1}(D_r)$ is a proper subplane of Π_1, i.e. its order $k_r < \sqrt{q}$, and $r \mid \sqrt{q} - k_r$. If $k_r^2 = \sqrt{q}$ then $r \mid (q^{1/2} - q^{1/4}, q - 1)$ so that $r \leq q^{1/4} - 1$, violating $r^2 - r + 2 > \sqrt{q}$. Otherwise by 2.3 we have $k_r^2 + k_r + 2 \leq \sqrt{q}$. By (12) we have $r - 1 \leq k_r$ so that $(r-1)^2 + (r-1) + 2 \leq \sqrt{q}$, again a contradiction. This completes the induction step, and (13) follows. Combining (12) and (13), we obtain

(14) $|\mathcal{P}_q| \geq 2$, where $\mathcal{P}_q$ is the set of primes $r \mid q - 1$ such that $r \equiv 1 \bmod 3$ and $r^2 - r + 2 \leq \sqrt{q}$.

Denying conclusion (iii) of 1.8, we assume for the remainder of the proof that $q < 10^6$. From factor tables (eg. [1,pp.844,845]) we quickly see that the only values of $q < 10^6$ satisfying (8), (14) are as listed in Table 7B. (Checking is facilitated by the fact that the factorizations of $\sqrt{q} - 1$, $\sqrt{q}$, $\sqrt{q} + 1$ occupy adjacent entries in the factor tables.)

q	$\mathcal{P}_q$	q	$\mathcal{P}_q$
181^2	7, 13	701^2	7, 13
337^2	7, 13	3^{12}	7, 13
379^2	7, 19	797^2	7, 19
419^2	7, 19	883^2	7, 13
547^2	7, 13	911^2	7, 13, 19
571^2	13, 19	937^2	7, 13

TABLE 7B

Suppose first that $q = 883^2$. Then (12), (13) imply that $\mathrm{Fix}_{\Pi_1}(D_7)$, $\mathrm{Fix}_{\Pi_1}(D_{13})$ are disjoint proper subplanes of Π_1. Hence k_7, $k_{13} \le \sqrt{883}$; furthermore (12) gives $k_7 \equiv 1 \bmod 7$, $k_{13} \equiv 12 \bmod 13$; $13 \mid (k_7^2 + k_7 + 1)$, $7 \mid (k_{13}^2 + k_{13} + 1)$ implies $k_7 \equiv 3$ or $9 \bmod 13$, $k_{13} \equiv 2$ or $4 \bmod 7$. Since $k_7 \ne 22$ by the Bruck-Ryser Theorem, we must have $k_7 = 29$, $k_{13} = 25$. Note that $D_7^\tau = D_7$, so that $\mathrm{Fix}_{\Pi_1}(D_7)$, $\mathrm{Fix}_{\Pi_1^\tau}(D_7)$ are two subplanes of order 29 interchanged by τ; they are disjoint, since they are fixed pointwise by P, P^τ respectively. Let Π_7 be the subplane generated by $\mathrm{Fix}_{\Pi_1}(D_7)$, $\mathrm{Fix}_{\Pi_1^\tau}(D_7)$. Now Π_7 does not meet $\mathrm{Fix}_{\Pi_1}(D_{13})$, and so Π_7 is a proper subplane of $\mathrm{Fix}(\gamma)$. Therefore its order m_7 satisfies $29^2 \le m_7 \le 883$. In particular Π_7 is a maximal subplane of $\mathrm{Fix}(\gamma)$, and since Π_7 is fixed pointwise by D_7 while $\mathrm{Fix}(\gamma)$ is not, we have $\mathrm{Fix}(D_7, \gamma) = \Pi_7$. This yields $7 \mid 883^2 - m_7$, from which we obtain $m_7 \ne 29^2$, and so $m_7 \ge 29^2 + 29$, $m_7 \in \{875, 882\}$. But τ induces an involutory collineation of Π_7, and so $m_7 \not\equiv 2 \bmod 4$ by [17,Thm.3.2]. Thus $m_7 = 875$ is a non-square, and so τ induces a homology on Π_7 with centre X and axis l, say, both of which are fixed by D_{13}. Let $X_0, X_1, \ldots, X_{875}$ be the points of l in Π_7, and let $l_0, l_1, \ldots, l_{870}$ be the lines of $\mathrm{Fix}_{\Pi_1}(D_7)$. None of $X_0, X_1, \ldots, X_{875}$ belongs to $\mathrm{Fix}_{\Pi_1}(D_7)$; for instance if $X_0 \in \mathrm{Fix}_{\Pi_1}(D_7)$, then because $X_0^\tau = X_0$ we would have X_0 in both $\mathrm{Fix}_{\Pi_1}(D_7)$ and $\mathrm{Fix}_{\Pi_1^\tau}(D_7)$, which is impossible. Therefore $l_0, l_1, \ldots, l_{870}$ pass through *distinct* points of l in Π_7, so we may assume that $l_j \cap l = X_j$, $j = 0, 1, \ldots, 870$. Since D_{13} acts on $\mathrm{Fix}_{\Pi_1}(D_7)$ without fixing any line, it follows that D_{13} acts on $\{X_0, X_1, \ldots, X_{870}\}$ without fixing any point. Now D_{13} fixes $X, X_{871}, X_{872}, \ldots, X_{875}$, so that $\mathrm{Fix}_{\Pi_7}(D_{13})$ is a subplane of order 4. From the action of τ on Π_7, we deduce that τ also induces a homology on $\mathrm{Fix}_{\Pi_7}(D_{13})$, contradicting the fact that $\mathrm{Fix}_{\Pi_7}(D_{13})$ has even order. Therefore $q \ne 883^2$.

Suppose that $q = 911^2$. Then D_{19} acts nontrivially on Π_1 (for otherwise $k_{13} \le \sqrt{911}$, $k_{13} \equiv 1 \bmod 13$, and $k_{13} \equiv 2$ or $4 \bmod 7$; this is impossible). Likewise D_{13} acts nontrivially on Π_1 (for otherwise $k_7 \le \sqrt{911}$, $k_7 \equiv 1 \bmod 7$, and $k_7 \equiv 7$ or $11 \bmod 19$; this is impossible). Then $k_{19} \le \sqrt{911}$, $k_{19} \equiv 18 \bmod 19$, and $k_{19} \equiv 3$ or $9 \bmod 13$; impossible. Hence $q \ne 911^2$.

The remaining ten cases in Table 7B are eliminated much more quickly: for some $r \in \mathcal{P}_q$, the necessary conditions on k_r prove to be inconsistent. □

8. Proof of Theorem 1.9

Suppose that we are given a counterexample. Consider the following subgroups and elements of G, as represented by matrices in SL$(2,q)$:

$$P = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathrm{GF}(q) \right\} \in \mathrm{Syl}_p(G),$$
$$\tau = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$D = \left\{ \mathrm{diag}(d, d^{-1}) : d \in \mathrm{GF}(q) \setminus \{0\} \right\}, \qquad |D| = \tfrac{1}{2}(q-1),$$
$$N = \mathrm{N}_G(P) = PD, \qquad N^\tau = \mathrm{N}_G(P^\tau) = P^\tau D.$$

Mimicking the proof of 1.8, we obtain

(1′) $q > 3$;

(2′) G acts irreducibly on Π;

(3′) $\mathrm{Fix}(PD) = \emptyset$;

(4′) $\mathrm{Fix}(P)$ is a subplane of order q.

Note that Theorem 6.2 applies in view of Lemma 3.1(ii).

(6′) D acts faithfully on $\mathrm{Fix}(P)$.

For if $g \in D \setminus 1$ acts trivially on $\mathrm{Fix}(P)$ then $\langle g^\tau \rangle = \langle g \rangle$ implies $\mathrm{Fix}(P) = \mathrm{Fix}(g) = \mathrm{Fix}(g^\tau) = \mathrm{Fix}(P^\tau)$, contrary to (2′).

(8′) $q \equiv 3 \bmod 4$.

For otherwise D contains an involution γ and we conclude as in (7) that q is a square.

(10′) D has no orbit of length 3 on the points (or lines) of Fix(P).

The proof of (10′) is a much shorter variation of the proof of (10), since q is not a square.

(7′) τ is a Baer collineation of Π.

For suppose that τ is a homology of Π. If $g \in D \setminus 1$ then $g = g\tau \cdot \tau$ is the product of two involutions, so by 2.6, g is a generalized perspectivity of Π, and in fact a generalized homology since $|\langle g\rangle| \mid \frac{1}{2}(q-1)$. But by (3′), (10′) and 2.1 we have $\mathrm{Fix}(P, g) = \emptyset$. Thus D acts semiregularly on the points of Fix(P) and $|D| = \frac{1}{2}(q-1) \mid q^2+q+1$ so that $q \in \{3, 7\}$, $|D| \in \{1, 3\}$ contrary to (3′), (10′). This gives (7′).

(8″) $q \equiv 3 \bmod 8$.

For otherwise (8′) gives $q \equiv 7 \bmod 8$ and G contains an element g such that $g^2 = \tau$, violating Lemma 2.5(ii) of [28].

Again as in the proof of 1.8 we obtain

(11′) If $1 \neq H \leq D$ then Fix(PH) is either empty or a proper subplane of Fix(P).

Let D_r be the Sylow r-subgroup of D, for each prime $r \mid \frac{1}{2}(q-1)$.

(12′) Fix(PD_r) is a proper subplane of Fix(P) whenever $3 \neq r \mid \frac{1}{2}(q-1)$. The order k_r of this subplane satisfies $r \mid k_r - 1$.

If $\mathrm{Fix}(PD_{p_1}D_{p_2}\cdots D_{p_e}) \neq \emptyset$ for some distinct primes $p_1, p_2, \ldots, p_e$ dividing $\frac{1}{2}(q-1)$ then $\mathrm{Fix}(PD_{p_1}D_{p_2}\cdots D_{p_e})$ is a subplane which we call $\Pi_{p_1,p_2,\ldots,p_e}$ of order $k_{p_1,p_2,\ldots,p_e}$. We factorise $D = D_0D_1D_3$ where

$$D_1 = \prod_{r\in\mathcal{P}_q} D_r, \qquad D_0 \cap D_1D_3 = 1,$$

$$\mathcal{P}_q = \big\{r \mid \tfrac{1}{2}(q-1) : r \text{ is a prime}, r \equiv 1 \bmod 3, r^2 - r + 2 \leq q\big\}.$$

Once again imitating the proof of 1.8 we have

(13′) $\mathrm{Fix}(PD_1) = \emptyset$;

(14′) $|\mathcal{P}_q| \geq 2$.

By (14′), (8″) and [1,pp.844–853], q is one of 547, 11^3=1331, 1483, 2003, 2731, 3011, 3907, 4219, 4523, 4691. In each case $|\mathcal{P}_q| = 2$ and so $r \mid k_s^2+k_s+1 \leq q-1$, $s \mid k_r^2+k_r+1 \leq q-1$, $r \mid k_r-1$, $s \mid k_s-1$ where $\mathcal{P}_q = \{r, s\}$. This narrows the possibilities to those given in Table 8A.

q	$\frac{1}{2}(q-1)$	r	k_r	s	k_s
3011	5·7·43	7	36	43	44
3907	3^2·7·31	7	36	31	32
4523	7·17·19	7	64	19	39 or 58

Table 8A

If $q = 3011$ then $5 \nmid k_7^2 + k_7 + 1$ and so $\mathrm{Fix}(PD_7D_5) \neq \emptyset$, $5 \mid k_7 - k_{5,7}$, $k_{5,7} \neq 6$ so that $\Pi_{5,7} = \Pi_7$. But D_5 also fixes some point of Π_{43} so that $\Pi_7 \subsetneq \Pi_5$, $36^2 \leq k_5 < \sqrt{3011}$, a contradiction.

If $q = 3907$ then $3 \nmid k_7^2 + k_7 + 1$ and so $\mathrm{Fix}(PD_7D_3) \neq \emptyset$; $31 \mid k_{3,7}^2 + k_{3,7} + 1$, $3 \mid k_7 - k_{3,7}$ yields $k_{3,7} = 36$, $\Pi_7 \subseteq \Pi_3$. Since $k_7^2 > \sqrt{q}$ we have $\Pi_7 = \Pi_3$; but D_{17} fixes some point of Π_{31}, a contradiction. The same argument eliminates the case $q = 4523$. □

9. Proof of Theorems 1.3,5

The following assertions, (15) through (17), pertain to the proofs of Theorems 1.3 and 1.5(a),(b). Let τ, τ', Z_τ, μ, Q, P be as in §4 for $G \cong \mathrm{PSL}(3, q)$, or as in §5 for $G \cong \mathrm{PSU}(3, q)$.

We first suppose that τ is a homology of Π. If τ, τ' have the same centre X and axis l, then (X,l) is invariant under $\langle \mathrm{C}_G(\tau), \mathrm{C}_G(\tau')\rangle = G$ and so G consists of (X,l)-homologies of Π. Since $|G| \not| \, q^4-1$ this cannot occur. However τ and τ' commute, so they must have distinct centres and axes (see Prop. 2.4(i) of [28]). By [23,Thm.C(i),(iii)] the remaining conclusions follow. Therefore we may assume that

(15) $\quad$ Fix(τ) is a subplane of order q^2,

and derive a contradiction. Let K_τ denote the kernel of the action of $\mathrm{C}_G(\tau)$ on Fix(τ), and for $H \le \mathrm{C}_G(\tau)$ let $\overline{H}$ denote its image in $\overline{\mathrm{C}_G(\tau)} = \mathrm{C}_G(\tau)/K_\tau$, so that $\overline{H}$ is the collineation group induced by H on Fix(τ). By Fix$(\overline{H})$ we shall mean the substructure consisting of all points and lines of Fix(τ) which are fixed by $\overline{H}$, i.e. Fix$(\overline{H}) =$ Fix(τ, H). We show that

(16) $\quad$ $K_\tau \cap \mathrm{C}_G(\tau)' = \langle \tau \rangle$, i.e. $\langle \tau \rangle \subseteq K_\tau \subseteq Z_\tau$; and

(17) $\quad$ G acts irreducibly on Π.

(Observe the equivalence of the two formulations of (16): if $K_\tau \cap \mathrm{C}_G(\tau)' = \langle \tau \rangle$ then by considering quotients in $\mathrm{C}_G(\tau)/Z_\tau \cong \mathrm{PGL}(2,q)$ we obtain $K_\tau \subseteq Z_\tau$. The converse is immediate.)

Assume first that $q > 3$. We suppose that (16) fails. Since $\langle \tau \rangle \subsetneq K_\tau \cap \mathrm{C}_G(\tau)' \trianglelefteq \mathrm{C}_G(\tau)'$ and $\mathrm{C}_G(\tau)'/\langle \tau \rangle \cong \mathrm{PSL}(2,q)$ is simple, we have $K_\tau \supseteq \mathrm{C}_G(\tau)'$. By 4.1(vi), 5.1(vi), 5.4 there exist involutions $\tau_1, \tau_2 \in G$ such that $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)'\rangle = G$ and $\mathrm{C}_G(\tau_1)' \cap \mathrm{C}_G(\tau_2)'$ contains some $g \ne 1$. Now Fix(g) is a Baer subplane of Π fixed pointwise by $\langle \mathrm{C}_G(\tau_1)', \mathrm{C}_G(\tau_2)'\rangle = G$, contrary to 2.4. This gives (16). If $G \cong \mathrm{PSL}(3,q)$ then $\mathrm{C}_G(\tau)$ acts irreducibly on Fix(τ) by 4.1(v) and Theorem 1.8, which yields (17). If $G \cong \mathrm{PSU}(3,q)$, $q \ne 5$, 9 then $\overline{\mathrm{C}_G(\tau)'} \cong \mathrm{PSL}(2,q)$ acts irreducibly on Fix(τ) by Theorem 1.7, again yielding (17). For $G \cong \mathrm{PSU}(3,5)$ or PSU(3,9) we have $|\overline{Z_\tau}|$ divides $(q+1)/2\mu$, $\mu = (q+1,3)$; but if $\overline{Z_\tau} = 1$ then $\overline{\mathrm{C}_G(\tau)} \cong \mathrm{PGL}(2,q)$ acts irreducibly on Fix(τ) by Theorem 1.8, which yields (17). We may therefore assume that $G \cong \mathrm{PSU}(3,9)$, $|\overline{Z_\tau}| = 5$, and note that Fix$(\overline{Z_\tau})$

is invariant under $\overline{\mathrm{C}_G(\tau)}$. If $\mathrm{Fix}(\overline{Z_\tau})$ is a subplane of $\mathrm{Fix}(\tau)$ (necessarily proper since $\overline{Z_\tau} \leq \mathrm{Aut\,Fix}(\tau)$) then $\mathrm{Fix}(\overline{Z_\tau})$ is of order 4 or 9 by Theorem 3.3, which is impossible since $5 \not| \, 9^2 - 4$, $5 \not| \, 9^2 - 9$. Otherwise $\overline{Z_\tau}$ induces a generalized homology group of order 5 on $\mathrm{Fix}(\tau)$ and $\overline{\mathrm{C}_G(\tau)}'$ fixes an antiflag in $\mathrm{Fix}(\tau)$, contrary to Theorem 1.7.

Now suppose that $q = 3$. Since (17) is included in the hypothesis for $G \cong \mathrm{PSL}(3,3)$, we prove (17) for $G \cong \mathrm{PSU}(3,3)$. If G fixes a line l of Π then let w be the number of orbits of G on the points of l and let $G_1, G_2, \ldots, G_w$ be the respective stabilizers of point representatives from these orbits. Using (9) of [28] we compute $F_\nu(\tau)$, the number of points of l fixed by τ in the ν-th orbit, $\nu = 1, 2, \ldots, w$. Since $[G\!:\!G_\nu] \leq 82$, Mitchell's list [26,p.241] restricts such G_ν to be among the types listed in Table 9A.

Type	Type in Mitchell's list	$\lvert G_\nu \rvert$	$[G\!:\!G_\nu]$	$F_\nu(\tau)$	G_ν
1	—	6048	1	1	G
2	2	96	63	7	$\mathrm{C}_G(\tau)$
3	3	96	63	3	$\mathrm{N}_G(\langle \tau, \tau' \rangle)$
4	9	168	36	12	$\mathrm{PSL}(2,7)$
5	1	216	28	4	$\mathrm{N}_G(Q)$
6	—	108	56	8	subgroup of $\mathrm{N}_G(Q)$

TABLE 9A

If l contains n_i point orbits of type i, $i = 1, 2, \ldots, 6$ then

$$\sum_{\nu=1}^{w} [G\!:\!G_\nu] = n_1 + 63(n_2 + n_3) + 36n_4 + 28(n_5 + 2n_6) = 82,$$

$$\sum_{\nu=1}^{w} F_\nu(\tau) = n_1 + 7n_2 + 3n_3 + 12n_4 + 4(n_5 + 2n_6) = 10,$$

which has no simultaneous solution in non-negative integers $\{n_i\}$. By contradiction, this proves (17).

Suppose that (16) fails for $q = 3$. Then $K_\tau \supseteq \mathrm{O}_2(\mathrm{C}_G(\tau))$, a quaternion group of order 8. Now τ' induces a collineation of order at most 2 on $\mathrm{Fix}(\tau)$, so τ' fixes a point X of $\mathrm{Fix}(\tau)$. By 5.1(ix) we may choose an involution $\tau'' \in \mathrm{C}_G(\tau)$ such that $\tau'\tau'' \in \mathrm{O}_2(\mathrm{C}_G(\tau))$, $[\tau', \tau''] \neq 1$ so that τ'' also fixes X. But then X is fixed by $\langle \mathrm{O}_2(\mathrm{C}_G(\tau')), \mathrm{O}_2(\mathrm{C}_G(\tau''))\rangle = G$ by 5.1(viii), contrary to (17). This completes the proof of (16).

Proof of Theorem 1.3 (concluded), in which $G \cong \mathrm{PSL}(3, q)$. We may suppose that τ, P_0, P_1 are as in §4.

(18) $\mathrm{Fix}(\mathrm{C}_G(\tau)P_0)$, $\mathrm{Fix}(\mathrm{C}_G(\tau)P_1)$ are *not both* empty.

For suppose that $\mathrm{Fix}(\mathrm{C}_G(\tau)P_0) = \mathrm{Fix}(\mathrm{C}_G(\tau)P_1) = \emptyset$. Clearly $\mathrm{Fix}(P_i)$ is neither empty nor a triangle, so by 2.1, P_i is planar, $i = 0, 1$. Now τ does not induce a homology on $\mathrm{Fix}(P_i)$; otherwise (since $P_i\langle\tau\rangle \lhd \mathrm{C}_G(\tau)P_i$) its centre would be fixed by $\mathrm{C}_G(\tau)P_i$. Hence $\mathrm{Fix}(\tau, P_i)$ is a subplane, $i = 0, 1$. By (16), $\mathrm{C}_G(\tau)'$ induces $\mathrm{PSL}(2, q)$ on $\mathrm{Fix}(\tau)$, leaving invariant the subplanes $\mathrm{Fix}(\tau, P_0)$, $\mathrm{Fix}(\tau, P_1)$. If $q \notin \{5, 9\}$ then the latter two subplanes of $\mathrm{Fix}(\tau)$ are disjoint by 4.2(i), violating Corollary 5.2(iv) of [28]. Indeed the same contradiction is obtained for $q \in \{5, 9\}$. (Clearly the orders of $\mathrm{Fix}(P_0)$, $\mathrm{Fix}(\tau, P_0)$ are divisible by p; in particular $\mathrm{Fix}(\tau, P_0)$ is not of order 4. By Theorem 3.3, the additional hypothesis required in [28,Cor.5.2] is satisfied.) This gives (18).

By 4.2(ii),(iii) we may assume that X is a point of Π such that $G_X = \mathrm{C}_G(\tau)P_0$, $|X^G| = q^2 + q + 1$. By (17) the points of X^G are not collinear.

(19) X^G is not an arc.

For suppose that X^G is an arc. Clearly P_0 fixes $q+1$ points of X^G. (This is evident from the proof of 4.2(iv), in which P_0 fixes exactly $q+1$ points of Σ.) Therefore P_0 is planar, and since $\mathrm{C}_G(\tau)$ acts transitively on $P_0 \setminus 1$ by conjugation, Theorem 6.2 shows that $\mathrm{Fix}(g)$ is a Baer subplane, given any $g \in P_0 \setminus 1$. But $\mathrm{Fix}(g)$ contains only $q+1$ points of X^G, so let $Y \in X^G$ be outside $\mathrm{Fix}(g)$ and let l be the unique line of $\mathrm{Fix}(g)$ containing Y. Then l carries $p \geq 3$ points of X^G, which gives (19). Therefore 4.2(iv) yields

(20) G leaves invariant a Desarguesian subplane Π_0 of order q, on which G acts faithfully.

If X is the centre of the homology induced by τ on Π_0, then $\mathrm{C}_G(\tau)$ acts on $\mathrm{Fix}(\tau)$, fixing X, contrary to 4.1(v) and Theorem 1.8. □

Proof of Theorem 1.5(a) concluded,, in which $G \cong \mathrm{PSU}(3,q)$, $q \neq 5,\ 11$. We have $\mathrm{Fix}(\overline{g}) = \mathrm{Fix}(\overline{\tau'})$ for all $g \in K_{\tau'} \setminus 1$, since $\mathrm{Fix}(g) = \mathrm{Fix}(\tau')$. Also since $Z_{\tau'} \cap K_\tau \subseteq Z_{\tau'} \cap Z_\tau = 1$ by (16), we have $\overline{Z_{\tau'}} \cong Z_{\tau'} \cong Z_\tau$ is cyclic of order $(q+1)/\mu$, and $\overline{K_{\tau'}} \cong K_{\tau'} \cong K_\tau$. In particular $\overline{\tau'} \neq 1$.

(21) $$K_\tau = \langle \tau \rangle, \qquad |\overline{Z_\tau}| = \frac{q+1}{2\mu} > 1.$$

If $\overline{\tau'}$ is a Baer involution of $\mathrm{Fix}(\tau)$ then using 2.4 and (16), $|\overline{K_{\tau'}}| = |K_{\tau'}| = |K_\tau|$ divides $\big(q(q-1), (q+1)/\mu\big) = 2$, which yields (21).

Otherwise $\overline{\tau'}$ is a homology of $\mathrm{Fix}(\tau)$. Let $e = |K_\tau|$ and suppose that $e > 2$. By 5.1(vii) we may suppose that $\tau',\ \tau'' \in \mathrm{C}_G(\tau)$ are involutions such that $\langle K_{\tau'}, K_{\tau''}\rangle \supseteq \mathrm{C}_G(\tau)'$. Now a point X of $\mathrm{Fix}(\tau)$ common to the axes of $\overline{\tau'}$, $\overline{\tau''}$ is fixed by $\langle \overline{K_{\tau'}}, \overline{K_{\tau''}}\rangle \supseteq \overline{\mathrm{C}_G(\tau)'} \cong \mathrm{PSL}(2,q)$, contrary to Theorem 1.7. (Recall that the exceptional cases (iii), (iv) of 1.7 do not occur if $\mathrm{PSL}(2,q)$ contains involutory homologies.) This concludes the proof of (21).

(22) $\mathrm{Fix}(\overline{g})$ is not a subplane of $\mathrm{Fix}(\tau)$, for any $\overline{g} \in \overline{Z_\tau} \setminus 1$.

For suppose that $\mathrm{Fix}(\overline{g})$ is a (necessarily proper) subplane of $\mathrm{Fix}(\tau)$ for some $\overline{g} \in \overline{Z_\tau} \setminus 1$. Since $\mathrm{Fix}(\overline{g})$ is invariant under $\overline{\mathrm{C}_G(\tau)}$, Corollary 5.2(v) of [28] implies that $|\overline{Z_\tau}| = (q+1)/2\mu$ divides $q(q-1)$, i.e. $q \in \{3,\ 5,\ 11\}$. (If $q = 9$ then $|\overline{Z_\tau}| = 5 \nmid 81 - 4$ so that $\mathrm{Fix}(\overline{g})$ is not of order 4; by Theorem 3.3, the additional hypothesis required in [28,Cor.5.2] is satisfied.) By hypothesis this means that $q=3$, $\overline{\mathrm{C}_G(\tau)'} \cong \mathrm{A}_4$, $\mathrm{Fix}(\tau)$ is either Desarguesian or a Hughes plane of order 9 (see Prop. 2.7 of [28]), and $\overline{\mathrm{C}_G(\tau)}$ leaves invariant an oval $\mathcal{O}$ (i.e. quadrangle) of the subplane $\mathrm{Fix}(\overline{Z_\tau})$ of order 3. The group of all collineations of $\mathrm{Fix}(\tau)$ leaving $\mathcal{O}$ invariant is isomorphic to $\mathrm{S}_4 \times H$, where H fixes $\mathrm{Fix}(\overline{g})$ pointwise and $H \cong \mathrm{C}_2$ or S_3 according as $\mathrm{Fix}(\tau)$ is Desarguesian or Hughes. (See [32], [24,Cor.5,6] for the collineation groups of the Hughes planes.) In neither case does $\mathrm{S}_4 \times H$ contain a subgroup isomorphic to $\overline{\mathrm{C}_G(\tau)} = \mathrm{C}_G(\tau)/\langle\tau\rangle$. (Lemma 5.1(iii) yields $\overline{\mathrm{C}_G(\tau)} \cong \mathrm{A}_4 \rtimes \langle\theta\rangle$ where θ is an automorphism of A_4 of order 4. Any subgroup of $\mathrm{S}_4 \times H$ of order 48 is isomorphic to $\mathrm{S}_4 \times \mathrm{C}_2$ or $\mathrm{D}_8 \times \mathrm{S}_3$, where D_8 is dihedral of order 8. However, since θ permutes regularly the four Sylow 3-subgroups of A_4, it is easily seen that $\overline{\mathrm{C}_G(\tau)}$ has no subgroup isomorphic to S_3.)

(23) $\overline{Z_\tau}$ acts semiregularly on the points and lines of $\mathrm{Fix}(\tau)$.

For if $q=9$, (21) gives $|\overline{Z_\tau}|=5$ and by (22) it is clear that $\overline{Z_\tau}$ is a generalized homology group of the subplane $\mathrm{Fix}(\tau)$ of order 81. Since $\overline{\mathrm{C}_G(\tau)'}$ leaves invariant $\mathrm{Fix}(\overline{Z_\tau})$, it fixes at least an antiflag in $\mathrm{Fix}(\tau)$, contrary to Theorem 1.7.

Otherwise $q \neq 5, 9$ and Theorem 1.7 implies that $\overline{\mathrm{C}_G(\tau)'}$ fixes no point or line of $\mathrm{Fix}(\tau)$. If (23) is false then by (22) and 2.1, $\mathrm{Fix}(\overline{g})$ is a triangle invariant under $\overline{\mathrm{C}_G(\tau)'}$, for some $\overline{g} \in \overline{Z_\tau} \setminus 1$. But then Theorem 1.7 gives $q=3$, and (21) gives $|\overline{Z_\tau}|=2$ so that $\overline{g}$ is an involution, which can never be triangular. Therefore (23) must hold.

Now $|\overline{Z_\tau}| = (q+1)/2\mu$ divides $q^4+q^2+1 = (q^2-1)(q^2+2)+3$, and since $q \neq 5$ by hypothesis, we have $q=17$. By Theorem 1.9 and duality, we may assume that $\overline{\mathrm{C}_G(\tau)'}$ has a point orbit $\mathcal{O} \subset \mathrm{Fix}(\tau)$ which is an 18-arc. Let $\overline{h} \in \overline{\mathrm{C}_G(\tau)'}$ have order 17, and let $X \in \mathcal{O}$ be the unique point of $\mathcal{O}$ fixed by $\overline{h}$. We have $\overline{Z_\tau} = \langle\overline{g}\rangle \cong \mathrm{C}_3$, and (23) implies that $\{X, X^{\overline{g}}, X^{\overline{g}^2}\}$ is a triangle. But $\overline{h}$ fixes $X, X^{\overline{g}}, X^{\overline{g}^2}$ and so $\overline{h}$ is a Baer collineation of $\mathrm{Fix}(\tau)$. But $\overline{h}$ acts transitively on $\mathcal{O} \setminus \{X\}$ which is a 17-arc in $\mathrm{Fix}(\tau)$, whereas any point orbit of a Baer collineation consists of collinear points, a contradiction. □

Proof of Theorem 1.5(b) concluded,, in which $G \cong \mathrm{PSU}(3,q)$, $q \in \{5,\ 11\}$. We have

$$(24) \qquad \mathrm{Fix}\big(\mathrm{N}_G(Q)\big) \neq \emptyset.$$

For suppose that $\mathrm{Fix}\big(\mathrm{N}_G(Q)\big) = \emptyset$. Clearly $\mathrm{Fix}(Q)$ is neither empty nor a triangle, so by 2.1, $\mathrm{Fix}(Q)$ is a subplane of Π. By Theorem 6.1, $[Q\,{:}\,Q_0] < n_P \leq q^2$ where n_P is the order of $\Pi_P = \mathrm{Fix}(P)$ and Q_0 is the kernel of the action of Q on Π_P. Thus $P \subsetneq Q_0 \lhd \mathrm{N}_G(Q)$, so by 5.2(ii) we have $Q_0 = Q$, i.e. $\mathrm{Fix}(Q) = \Pi_P$. Let $y \in Q \setminus P$, and let n_P, n_y be the respective orders of the subplanes Π_P, $\mathrm{Fix}(y)$. Since q is prime, P acts semiregularly on the set of points of l outside Π_P, where l is a given line of Π_P, so that $q \,\big|\, q^4 - n_P$, i.e. $q \,\big|\, n_P$. If $n_y = n_P$ then Q acts semiregularly on the points of l outside Π_P, and $q^3 \,\big|\, q^4 - n_P$, contradicting $n_P \leq q^2$. Hence $q^2 \leq n_P^2 \leq n_y \leq q^2$, and we have equality: $n_P = q$, $n_y = q^2$.

Since Q acts trivially on Π_P, and by 5.2(i), 5.4, the collineation group $\overline{N}$ (say) induced by $\mathrm{N}_G(Q)$ on Π_P is cyclic of order dividing $(q^2-1)/\mu$. If $q = 5$ then $|\overline{N}| \,\big|\, 8$ and clearly $\overline{N}$ fixes a point of Π_P. Hence we may assume that $q = 11$, $|\overline{N}| \,\big|\, 40$. If $2 \,\big|\, |\overline{N}|$ then $\overline{N}$ contains a homology of Π_P, whose centre is fixed by $\overline{N}$. Otherwise $|\overline{N}| \,\big|\, 5$ and $\overline{N}$ fixes at least 3 of the $11^2 + 11 + 1$ points of Π_P. This proves (24).

By (24) and duality we may assume that G has a point orbit $\mathcal{O}$ of length $q^3 + 1$. By 5.3(ii) and (17), $\mathcal{O}$ is either an oval or a unital embedded in Π. But if $\mathcal{O}$ is a unital then τ fixes exactly $q+1$ collinear points of $\mathcal{O}$, whose common line is fixed by $\mathrm{C}_G(\tau)$, and so $\mathrm{C}_G(\tau)'$ acts reducibly on the subplane $\mathrm{Fix}(\tau)$, and by 1.7 we have $q \neq 11$ in this case. □

Proof of Theorem 1.5(c), in which $G \cong \mathrm{PGU}(3,q)$. Mimicking the proof of 1.5(a), it is clear that

$$(21') \qquad K_\tau = \langle \tau \rangle, \qquad |\overline{Z_\tau}| = \tfrac{1}{2}(q+1) > 1$$

and that (22) holds. If $q \neq 5$ then (23) holds and $|\overline{Z_\tau}| = \frac{1}{2}(q+1)$ divides $q^4 + q^2 + 1$, a contradiction as before. For the remainder of the proof we may therefore assume that

$q = 5$, $|\overline{Z_\tau}| = 3$. Consider the action of $\overline{\mathrm{C}_G(\tau)'} \cong \mathrm{PSL}(2,5)$ on the subplane $\mathrm{Fix}(\tau)$ of order 25.

Suppose that $\overline{\mathrm{C}_G(\tau)'}$ acts reducibly on $\mathrm{Fix}(\tau)$. By Theorem 1.7, $\overline{\mathrm{C}_G(\tau)'}$ fixes an antiflag (X, l) and has orbits $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ of length 5, 5, 6, 10 on l, respectively. Now $\overline{Z_\tau}$ fixes all 26 points of l in $\mathrm{Fix}(\tau)$. (For $\overline{Z_\tau}$ cannot interchange $\mathcal{O}_1$ and $\mathcal{O}_2$ since $|\overline{Z_\tau}| = 3$. Thus $\overline{Z_\tau}$ leaves each $\mathcal{O}_i$ invariant, $i = 1, 2, 3, 4$. If $Y \in \mathcal{O}_i$ then Y is the unique point of $\mathcal{O}_i$ fixed by $\overline{\mathrm{C}_G(\tau)'}$, so that $\overline{Z_\tau}$ fixes Y as claimed.) Let $\overline{g} \in \overline{\mathrm{C}_G(\tau)'}$ be an involution. Then $\overline{g}$ fixes 1, 1, 2, 2 points of $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ respectively, so that $\mathrm{Fix}(\overline{g})$ is a subplane of order 5 on which $\overline{Z_\tau}$ induces an (X, l)-homology group, contradicting $|\overline{Z_\tau}| = 3$.

Hence $\overline{\mathrm{C}_G(\tau)'}$ acts irreducibly on $\mathrm{Fix}(\tau)$, and by duality we may assume by Theorem 1.9 that $\overline{\mathrm{C}_G(\tau)'}$ has a point orbit $\mathcal{O}$ which is a 6-arc. This leads to a contradiction just as in the case PSU(3,17) treated above. □

10. The Case PSL(3,3)

We indicate here what happens when the additional hypothesis in Theorem 1.3 for $q = 3$ is removed. Suppose that Π is a projective plane of order 81 admitting a *reducible* collineation group $G \cong \mathrm{PGL}(3,3) = \mathrm{PSL}(3,3)$. We claim that $\mathrm{Fix}(G)$ is a subplane of order 3. To see this, suppose that G fixes a line l, and let n_1 be the number of points of l fixed by G. For every maximal subgroup H of G satisfying $[G : H] \leq 82$ we have $[G : H] = 13$ (type 1 or 2 in the list of Mitchell [26,p.241]) so that $82 \equiv n_1 \mod 13$. However $n_1 \leq 10$ by (15), and so $n_1 = 4$. Dually, every point of Π fixed by G lies on exactly 4 fixed lines. Hence $\mathrm{Fix}(G)$ is a subplane of order 3 as claimed.

For the remainder of this paper we assume that Π is a projective plane of order q^4 admitting $G \cong \mathrm{PGL}(3, q)$ fixing *pointwise* a subplane Π_0 of order q; having $q^2 + q + 1$ point orbits of length $q^4 - q$ (those points outside Π_0 but on some line of Π_0); and with the remaining $q^3(q^3 - 1)(q^2 - 1)$ points of Π forming a regular G-orbit. These conditions are satisfied by the Lorimer-Rahilly translation plane in case $q = 2$. By Theorem 1.3 (see assertion (17), which is also implicit in the statement of Theorem 1.3) the only permissible odd value of q is 3. The case $q=3$ cannot yield a translation plane by [20] or [21,Lemma 4.6]; nevertheless to settle this exceptional possibility is a very interesting problem in which the case $q=2$ provides some inspiration. We proceed with the highlights, omitting the details.

In any case by our hypothesis there exist flags (X,l), (Y,m) such that the stabilizers $G_X=G_l=1$, and $G_Y=G_m$ of order $q^2(q^2-1)$. Then G_Y acts *tangentially transitively* on Π relative to the Baer subplane Fix(G_Y) (in a slight and obvious extension of the terminology of Jha [20]). Furthermore $\mathrm{N}_G(G_Y)/G_Y$ is a group of order $q(q-1)$ acting faithfully on Fix(G_Y), tangentially transitively relative to $\Pi_0=$ Fix(G). If $\{g_1,g_2,\ldots,g_m\}$ is a set of $m=q^2+q+1$ representatives of the distinct right cosets of $\mathrm{N}_G(G_Y)$ in G, then we may express G as the *disjoint* union

$$G=\{1\}\cup\big(G_Y^{g_1}\setminus 1\big)\cup\big(G_Y^{g_2}\setminus 1\big)\cup\cdots\cup\big(G_Y^{g_m}\setminus 1\big)\cup T$$

where T is a normal *subset* of size $q(q^3-q-1)\big[q(q^3-q-1)-1\big]$. If $S=\{g\in G: X^g\in l\}$ then $|S|=q(q^3-q-1)$ and every element $t\in T$ is representable uniquely as $t=s_1s_2^{-1}$ with $s_1,s_2\in S$. Also every $t\in T$ is representable uniquely as $t=s_3^{-1}s_4$ with $s_3,s_4\in S$. (Thus S is a sort of 'partial difference set'.)

In case $q=2$ with the Lorimer-Rahilly plane, we may identify $G\cong \mathrm{PGL}(3,2)$ with the permutation group $\big\langle(1234567),(12)(36)\big\rangle<\mathrm{A}_7$, and then (X,l) may be chosen such that $S=\big\{(1), (1264735), (1274653), (1367425), (1576423), (14)(3756), (16)(2437), (17)(2456), (34)(1752), (45)(1632)\big\}$. In this case $G_Y\cong \mathrm{A}_4$ and T is the union of the conjugacy classes of types 4A, 7A and 7B in the notation of the Atlas [3,p.3].

In case $q=3$ we have $G_Y\cong 3^2{:}\mathrm{Q}_8$ where Q_8 is quaternion of order 8. Also $\mathrm{N}_G(G_Y)/G_Y\cong \mathrm{S}_3$, so that Fix$(G_Y)$ is a Hughes, Hall or dual Hall plane of order 9 (as in the comments following Theorem 6.2). In the Atlas notation [3,p.13], T is the union of the conjugacy classes of types 3B, 6A, 8A, 8B, 13A, 13B, 13C, 13D. A crucial step in the construction of such an exceptional plane of order 81 appears to be finding a subset $S\subset G$ of size 69 satisfying the above 'partial difference set' condition, which remains an open problem.

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