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PSL$(2,q)$ AS A COLLINEATION GROUP OF PROJECTIVE PLANES OF SMALL ORDER[1]

Abstract. Let Π be a projective plane of order n admitting a collineation group $G \cong \mathrm{PSL}(2,q)$ for some prime power q. It is well known for $n = q$ that Π must be Desarguesian. We show that if $n < q$ then only finitely many cases may occur for Π, all of which are Desarguesian. We obtain some information in case $n = q^2$ with q odd, notably that G acts irreducibly on Π for $q \neq 3, 5, 9$.

1. Main Results

Let $q = p^m$, p a prime, $m \geq 1$ an integer.

A model for our results is the theorem of Lüneburg [16] and Yaqub [23] (see Theorem 2.10 below) which characterizes the Desarguesian projective plane of order q by a collineation group $G \cong \mathrm{PSL}(2,q)$. Other authors (see [9], [10], [15], [22]) have extended this result to planes of more arbitrary order, but at the expense of requiring two main additional assumptions: that G acts **irreducibly** on Π (i.e. fixing no point, line or triangle) and that G contains non-identity **perspectivities** of Π. We obtain some progress without such additional assumptions, for planes of suitably restricted order.

1.1 THEOREM. *If a projective plane Π of order $n < q$ admits a collineation group $G \cong \mathrm{PSL}(2,q)$, then Π is Desarguesian and $(n,q) = (2,3)$, $(2,7)$, $(4,5)$, $(4,7)$ or $(4,9)$. Moreover each of the latter cases indeed occurs.*

The essential reason why the argument of [18] does not suffice to prove 1.1 is that here, unlike the situation of Theorem 2.10, one cannot immediately assert that a Sylow p-subgroup of G fixes a flag of Π. To prove Theorem 1.1 (and 1.2 below) we first limit the possibilities for the point stabilizers using the classification of subgroups of G as given in [3] or [12], then solve by exhaustive case analysis a list of Diophantine equations expressing the numbers of points of Π fixed by distinguished elements of G such as involutions and p-elements. At the outset of the proofs in sections 3 and 4, we reduce to the case that involutions in G are Baer.

Consider now a projective plane Π of order q^2 admitting a collineation group $G \cong \mathrm{PSL}(2,q)$. For q even, a determination of all such planes Π appears to be very difficult, since G need not act irreducibly (in the sense defined above). (For example, see [5] for a list of translation planes of order q^2 admitting $\mathrm{PSL}(2,q) \cong \mathrm{SL}(2,q)$ for q even.) Henceforth we therefore suppose that q is odd. If Π is Desarguesian or a generalized Hughes plane of order q^2, then Π admits $G \cong \mathrm{PSL}(2,q)$ as a collineation group, G leaves invariant a subplane Π_0 of order q, and for $q > 3$ the collineation group G is irreducible. No other possibilities for Π

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are known, and our remaining results may be viewed as steps toward determining all such planes Π. (See [17] for a definitive description of the generalized Hughes planes, together with a characterization of such planes by the little projective group of Π_0.)

1.2 THEOREM. *Suppose that a projective plane* Π *of order* q^2 *admits a collineation group* $G \cong \mathrm{PSL}(2, q)$, *where* q *is odd. Then one of the following must hold:*

(i) G *acts irreducibly on* Π*;*

(ii) $q = 3$ *and* G *fixes a triangle but no point or line of* Π*;*

(iii) $q = 5$, $\mathrm{Fix}(G)$ *consists of an antiflag* (X, l), *and* G *has point orbits of length 5, 5, 6, 10 on* l*; or*

(iv) $q = 9$ *and* $\mathrm{Fix}(G)$ *consists of a flag.*

Note in case (ii) of the above that Π is either Desarguesian or a Hughes plane of order 9 (see Proposition 2.7). No occurrences of (iii) or (iv) are, however, known.

We obtain as a corollary (see 5.1) the following fact, proved independently by Foulser and Johnson [4] using representation-theoretic methods: any projective translation plane of order q^2 admitting $G \cong \mathrm{PSL}(2, q)$ for q odd is necessarily Desarguesian.

In [20] we apply these results to the situation where a projective plane Π of order q^4 admits a collineation group $G \cong \mathrm{PSL}(3, q)$ or $\mathrm{PSU}(3, q)$. If an involution $\tau \in G$ is Baer and $\langle \tau \rangle$ is the kernel of the action of $\mathrm{C}_G(\tau)' \cong \mathrm{SL}(2, q)$ on $\mathrm{Fix}(\tau)$, then Theorem 1.2 applies; here Corollary 5.2 is often helpful as well. In most cases a contradiction ensues and we conclude that G contains involutory homologies.

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2. Notation and Preliminaries

Most of our notation and terminology is standard and follows [2], [11] for projective planes; [6] for finite groups. We assume some of the better-known results, and in the following we omit proofs of the more accessible statements.

A pair (X, l) consisting of a point X and a line l of a projective plane, is a **flag** or an **antiflag** according as $X \in l$ or $X \notin l$. By (l) we mean the set of points on the line l; we write l in place of (l) if the intent is clear. An n-**arc** is a set of n points, no three of which are collinear. An **oval** is an $(n+1)$-arc in a projective plane of order n. A **hyperoval** is an $(n + 2)$-arc in a projective plane of order n; such can only exist for n even. If S is any set of collineations of a projective plane Π, we denote by $\mathrm{Fix}(S)$ the closed substructure of Π consisting of all points and lines fixed by every member of S. If $\mathrm{Fix}(G)$ is a subplane (respectively, a Baer subplane, a triangle) of Π, we say that G is a **planar** (resp., **Baer**, **triangular**) collineation group of Π.

A collineation $g \neq 1$ of Π is a **generalized** (X, l)**-perspectivity** if it fixes the point X and the line l, and if any additional fixed points (resp., lines) lie on l (resp., pass through X). If g fixes l pointwise and X linewise, g is an (X, l)-**perspectivity** with **centre** X and **axis** l. (Following [2], we include the identity collineation as both a perspectivity and a generalized perspectivity.) We say **elation** or **homology** in place of 'perspectivity' according as $X \in l$ or $X \notin l$. By $G(X,l)$ we denote the subgroup of G consisting of (X,l)-perspectivities of Π.

For a group G of permutations on a set Ω, we say that G acts **semiregularly** on Ω if the stabilizer $G_X = 1$ for all $X \in \Omega$; G acts **regularly** if it acts both semiregularly and transitively. The symmetric and alternating groups of degree n are denoted by S_n and A_n respectively. An **involution** is a group element of order 2.

We assume the reader's dexterity in applying the results of [3], [12], which classify the subgroups of PSL$(2,q)$. Note that if $G \cong \mathrm{PGL}(2,q)$ then G is isomorphic to a subgroup of PSL$(2,q^2)$, so by applying [3], [12] to PSL$(2,q^2)$ as well as to $G' \cong \mathrm{PSL}(2,q)$, we may in fact classify the subgroups of G.

2.1 PROPOSITION. *If G acts on a projective plane Π such that* $\mathrm{Fix}(G) = \emptyset$, *then for any $N \trianglelefteq G$,* $\mathrm{Fix}(N)$ *is either empty, a triangle, or a (not necessarily proper) subplane of* Π.

For a proof, see [8,Cor.3.6].

2.2 PROPOSITION. *If G is a Baer collineation group of a projective plane Π of order n^2, then $|G| \mid n(n-1)$.*

Proof. Let l be a line of $\mathrm{Fix}(G)$. Since $\mathrm{Fix}(G)$ is a maximal closed substructure of Π, G acts semiregularly on the $n(n-1)$ points of l outside $\mathrm{Fix}(G)$. □

2.3 THEOREM. *If G is an abelian planar collineation group of a projective plane Π of order n, then $|G| < n$.*

Proof. G acts faithfully on an orbit of maximal length (see [11,p.254]), which by duality we may assume to be a line orbit. Since G is abelian, this means that there is a line l whose stabilizer is $G_l = 1$. We may assume that l meets no point of $\Pi_G = \mathrm{Fix}(G)$; for if l contains a point X of Π_G then $|G| = |l^G| \le n - n_G$ where n_G is the order of Π_G, and we are done.

Let $l_1 \ne l_2$ be lines of Π_G and let the point $X_i = l \cap l_i$ have stabilizer G_i, $i = 1, 2$; we may assume that $|G_1| \le |G_2|$. Now $l = X_1X_2$ implies

$$G_1 \cap G_2 \subseteq G_l = 1; \qquad |G| \ge |G_1||G_2| \ge |G_1|^2.$$

Let Π_1 be the subplane generated by Π_G and X_1^G. Since G is abelian, G_1 fixes every point of X_1^G, so if $\Pi_1 = \Pi$ then G_1 acts trivially on Π, and $|G| = |X_1^G| \le n - n_G$. Otherwise Π_1 is a proper subplane of Π, of order n_1, say. Now

$$[G\!:\!G_1] = |X_1^G| \le n_1 - n_G < n_1 \le n^{1/2}$$

and so $|G| < n^{1/2}|G|^{1/2}$, which yields $|G| < n$. □

2.4 PROPOSITION.

(i) *If* g_i *is a nontrivial* (X_i, l_i)*-homology of* Π, $i = 1, 2$ *such that* g_1, g_2 *commute, then either* $(X_1, l_1) = (X_2, l_2)$ *or* $X_1 \in l_2$, $X_2 \in l_1$.

(ii) *Two commuting elations of* Π *have the same axis or the same centre.*

2.5 LEMMA. *Let* Π *be a projective plane of order* n^2 *where* $n \equiv 2$ *or* 3 *mod 4, and let* G *be a collineation group of* Π.

(i) *If* $\mathrm{Fix}(G) \neq \emptyset$ *then every involution in the derived subgroup* G' *is a perspectivity of* Π.

(ii) *If* G *is cyclic of order 4 then the involution in* G *is a perspectivity of* Π.

Proof of 2.5. Suppose that G fixes a line l of Π, and that G' contains a Baer involution τ. Then τ fixes exactly $n+1$ points of l and permutes the remaining $n(n-1)$ points of l in pairs, thereby inducing an odd permutation on (l), which is impossible. This proves (i). In (ii) it is clear that a generator of G fixes some line of Π, and the proof follows similarly. □

2.6 PROPOSITION. *Suppose that* Π *is a projective plane of order* n^2, Π_0 *a Baer subplane, and that* τ_0, τ_1 *are commuting involutory collineations of* Π *which leave* Π_0 *invariant, such that* τ_0, τ_1 *induce homologies of* Π_0 *with distinct axes. Then* $\langle \tau_0, \tau_1 \rangle$ *contains an involutory homology of* Π.

Proof. By 2.4 there exists a triangle with vertices X_0, X_1, X_2 in Π_0 such that τ_i induces an involutory $(X_i,\, X_{i+1}X_{i+2})$-homology of Π_0, $i = 0, 1, 2$ where $\tau_2 = \tau_0\tau_1$ and the subscripts are read modulo 3. Suppose that τ_0, τ_1, τ_2 are Baer involutions of Π. Since τ_0 induces an involutory collineation of $\mathrm{Fix}(\tau_1)$, the group $\langle \tau_0, \tau_1 \rangle$ fixes at least three points on some side of the triangle $X_0X_1X_2$. We may assume that $\langle \tau_0, \tau_1 \rangle$ fixes a point $X \in X_1X_2 \setminus \{X_1, X_2\}$. Now τ_0 fixes exactly $n+1$ lines through X_0, all of which belong to Π_0, so that X_0X is a line of Π_0. Therefore $X = X_0X \cap X_1X_2$ is a point of Π_0 fixed by $\langle \tau_0, \tau_1 \rangle$, a contradiction. □

2.7 PROPOSITION. *Let* Π *be a projective plane of order* n. *If* $n < 9$ *then* Π *is Desarguesian. If* $n = 9$ *and* Π *admits an involutory collineation, then* Π *is a Desarguesian, Hughes, Hall, or dual Hall plane. If* $n = 9$ *and* Π *admits a collineation group isomorphic to* A_4, *then* Π *is a Desarguesian or Hughes plane.*

Proof. For $n \leq 8$ see [2,p.144]; for $n = 9$ with an involution see [13]. That a Hall plane of order 9 does not admit A_4 follows from [14,Theorem 3.3]. □

2.8 LEMMA. *Let* $G \cong \mathrm{PSL}(2, q)$, q *odd,* N *the normalizer of a Sylow* p*-subgroup of* G, H *a subgroup of* N *such that* $|H| \geq \frac{1}{2}(q+1)$. *Then* $|H|$ *is either*

(i) qd *for some* $d \mid \frac{1}{2}(q-1)$, *or*

(ii) $p^{m-e}(p^e - 1)$ *where* $2e \mid m$.

Proof. We have the semidirect product $H = P \rtimes K$ where K is cyclic and P is elementary abelian of order p^l, $l \le m$. Now K acts semiregularly on P by conjugation, so that

$$|K| \,\big|\, (p^l - 1, p^m - 1) = p^e - 1 \qquad \text{where } e = (l, m).$$

We have

$$\tfrac{1}{2}p^m < |H| \le p^l(p^e - 1) < p^{l+e},$$

and so $m < l + e + 1$. Writing $m = m'e$, $l = l'e$, this gives $m' < l' + 2$. We may assume that $l' < m'$, so that $l' = m' - 1$, i.e. $l = m - e$. Now $|K| = p^e - 1$, for otherwise

$$\tfrac{1}{2}(p^m + 1) \le \tfrac{1}{2}p^l(p^e - 1) = \tfrac{1}{2}(p^m - p^l),$$

which is absurd. Finally, $|K| \,\big|\, \frac{1}{2}(q - 1)$ yields

$$m' = \frac{m}{e} \equiv \frac{p^m - 1}{p^e - 1} \equiv 0 \mod 2.$$

□

Remarks. We can do no better than the above; namely, such an H exists for each order permitted by the conclusion of 2.8. Note that case (ii) of 2.8 occurs precisely when q is a square.

2.9 PROPOSITION. *Suppose that* $\mathrm{PSL}(2, q) \le G \le \mathrm{PGL}(2, q)$ *or* $\mathrm{SL}(2, q) \le G \le \mathrm{GL}(2, q)$.

(i) A subgroup $H \le G$ satisfies $[G : H] = q + 1$ if and only if $H = \mathrm{N}_G(P)$ for some Sylow p-subgroup P of G.

(ii) Suppose that G acts (not necessarily faithfully) on a projective plane Π, and let X be a point of Π. If the orbit X^G has length $q + 1$, then its points are either collinear or form an arc.

We omit the proof of (i). To prove (ii) suppose that X^G contains three collinear points, say X, Y, Z. The group $G_{X,Y}$ either acts transitively on $X^G \setminus \{X, Y\}$ or has two orbits of length $\frac{1}{2}(q - 1)$ each (in which case q must be odd). We may suppose that the line XY contains exactly $2 + \frac{1}{2}(q - 1)$ points of X^G. Since G acts 2-transitively on X^G, its points form a 2-design (see [1]) with $\lambda = 1$, $v = q + 1$, $k = \frac{1}{2}(q + 3)$. But then $r = q/(k - 1) = 2q/(q + 1)$ is an integer, which is impossible. □

2.10 THEOREM (Lüneburg [16], Yaqub [23]). *Suppose that a projective plane Π of order q admits a collineation group $G \cong \mathrm{PSL}(2, q)$. Then Π is Desarguesian. Furthermore G acts irreducibly on Π for odd $q > 3$, and leaves invariant a triangle but no point or line for $q = 3$. G fixes a point and/or line of Π if q is even.*

For a proof, see [18].

3. Proof of Theorem 1.2

If $q = 3$ then Π is either Desarguesian or a Hughes plane of order 9 (by 2.7) and the result is immediate. We may therefore assume that

(1) $q > 3$, G is simple, G fixes a line l of Π,

and in the remainder of the proof we derive a contradiction. Now

(2) G does not fix l pointwise.

For otherwise the elations of Π in G with axis l form a normal subgroup N, where $N \neq 1$ since N includes the p-elements of G. But $N \neq G$ since the involutions in G are homologies, contradicting the simplicity of G. This proves (2).

G has a single conjugacy class of involutions. Suppose that these involutions are homologies of Π. Let $E = \langle \tau, \tau' \rangle$ where $\tau \neq \tau'$ are commuting involutions in G. If τ, τ' have a common axis l' then l' is fixed by $\langle \mathrm{C}_G(\tau), \mathrm{C}_G(\tau') \rangle = G$, and since all involutions are conjugate in G, this means that all involutions in G are homologies with axis l', contrary to (2). Otherwise by 2.4, the axes and centres of the involutions in E form a triangle, of which one side must be l, and this leads to the same contradiction with (2). This proves that

(3) the involutions in G are Baer collineations of Π, each fixing $q + 1$ points of l.

By Lemma 2.5(i) we have

(4) $q \equiv 1 \bmod 4$.

Let w be the number of point orbits of G on l, and let G_1, G_2, …, G_w be the respective stabilizers of the point representatives from these orbits. Let $F_\nu(\tau)$, $F_\nu(E)$ be the number of points of l fixed by τ, E respectively, in the ν-th orbit, $\nu = 1, 2, \ldots, w$. Then the following must hold:

(5) $\sum_{\nu=1}^{w} [G\!:\!G_\nu] = q^2 + 1;$

(6) $\sum_{\nu=1}^{w} F_\nu(\tau) = q + 1;$

(7) $\sum_{\nu=1}^{w} F_\nu(E) = 2$ or $q + 1$, or possibly $q^{1/2} + 1$ if q is a square.

In particular, since $|G| = \frac{1}{2}q(q^2 - 1)$, (5) gives

(8) $|G_\nu| \geq \frac{1}{2}(q + 1), \qquad \nu = 1, 2, \ldots, w.$

More generally, if H is a subset of G, then the number of points of l fixed by H in the ν-th orbit may be readily computed via

Type	G_ν	$[G:G_\nu]$	$F_\nu(\tau)$	$F_\nu(E)$
1	G	1	1	1
2	order qd, $d \mid \frac{1}{2}(q-1)$	$(q^2-1)/2d$	$\begin{cases}(q-1)/d, & d \text{ even}\\ 0, & d \text{ odd}\end{cases}$	0
4	dihedral of order $q-1$	$\frac{1}{2}q(q+1)$	$\frac{1}{2}(q+1)$	3
5	dihedral of order $q+1$	$\frac{1}{2}q(q-1)$	$\frac{1}{2}(q-1)$	0
6	cyclic of order $\frac{1}{2}(q+1)$	$q(q-1)$	0	0
11	A_4 (for $q =$ 5, 13 only)	$\frac{1}{24}q(q^2-1)$	$\frac{1}{4}(q-1)$	1
12	A_5 (for $q = 29$, 61, 101, 109 only)	$\frac{1}{120}q(q^2-1)$	$\frac{1}{4}(q-1)$	1

Table 3A

$$F_\nu(H) = \frac{|N_G(\langle H\rangle)|}{|G_\nu|} \cdot \big|\{U \le G_\nu : U \text{ is conjugate to } \langle H\rangle \text{ in } G\}\big|. \tag{9}$$

(For if $X \in l$ such that $G_X = G_\nu$ then $|G_X| \cdot F_\nu(H) = |\{g \in G : H \text{ fixes } X^g\}| = |\{g \in G : \langle H\rangle^{g^{-1}} \subseteq G_X\}| = |N_G(\langle H\rangle)| \cdot |\{U \le G_X : U \text{ is conjugate to } \langle H\rangle \text{ in } G\}|$.)

We now suppose that $q \equiv 5 \bmod 8$, so that in particular q is not a square. By [3], [12] and Lemma 2.8, the only possibilities for G_ν satisfying (8) are as listed in Table 3A.

We first verify the following:

(10) If $q \equiv 5 \bmod 8$, then the orbit structure of G on the points of l belongs to one of the following four cases:

(i) $q + 1$ fixed points and one point orbit of type 6;
(ii) 2 fixed points and the remaining point orbits of type 2;
(iii) types 1, 11, 5, 5 (having lengths 1, 5, 10, 10 respectively) for $q = 5$; or
(iv) types 11, 11, 5, 2 (having lengths 5, 5, 10, 6 respectively) for $q = 5$.

Let n_k denote the number of orbits of type k on the points of l. If $n_1 = q + 1$ then the remaining $q(q-1)$ points of l are not fixed by any involution, and so must comprise orbits of type 2 (with d odd) or type 6. But each orbit of type 2 has length divisible by $q + 1$, so we have case (10, i). Therefore in proving (10), we may assume that $n_1 \le q$. Also $n_6 = 0$, for otherwise we again have case (10, i). Now (7) refines to

$$\sum_{\nu=1}^{w} F_\nu(E) = 2,$$

for otherwise (since q is not a square) E would fix $q+1$ points of l, and these $q+1$ points would form a set invariant under $\langle \mathrm{C}_G(\tau), \mathrm{C}_G(\tau')\rangle = G$, but since all involutions are conjugate in G, they would all fix the same $q+1$ points of l, contradicting $n_1 \le q$. Now this simply says that

$$n_1 + n_{11} + n_{12} = 2, \qquad n_4 = 0,$$

and the remaining orbits are of type 2 or 5. If $n_1 = 2$, $n_{11} = n_{12} = 0$ then by (5) we have $n_5 \le 2$ and $\frac{1}{2}q(q-1)n_5 \equiv 0 \bmod (q+1)$ (since type 2 orbits are of length divisible by $q+1$), i.e. $n_5 \equiv 0 \bmod \frac{1}{2}(q+1)$, so that $n_5 = 0$ and we have case (10, ii). If $n_{11} > 0$ then $q = 5$ or 13; by (5), (6) we obtain cases (iii), (iv) of (10). If $n_{12} > 0$ then $q = 29$, 61, 101 or 109; by (5), (6) this is impossible. This concludes the proof of (10).

Next we prove that

(11) case (10, iii) cannot occur.

For suppose that case (10, iii) holds. We have $q = 5$, $G \cong \mathrm{A}_5$. Let $\sigma \in G$ be an element of order 3 which normalizes E, and let X, Y be the two points of l fixed by E. Since τ' induces a homology on the Baer subplane Fix(τ), we may assume that E fixes exactly two lines l, l_2 through X and exactly six lines l, l_3, l_4, …, l_7 through Y. Then σ fixes l_2 and at least two of l_3, l_4, …, l_7, as well as five points of l. Hence σ is planar.

Now let $\sigma' \in G$ be an element of order 3 such that $\tau\sigma'\tau = (\sigma')^2$. Then $\langle \sigma', \tau\rangle$ fixes exactly 3 points of l, so τ induces a Baer collineation of the subplane Fix(σ'). This means that the subplane Fix(τ) of order 5 contains a subplane Fix(σ', τ) of order 2, which is impossible by Bruck's Theorem [11,p.81]. This completes the proof of (11).

We may also assume that

(12) case (10, iv) does not occur.

For suppose that we have case (10, iv). If G fixes a point outside l then we have conclusion (iii) of 1.2. We may assume otherwise. Let the point orbits of G on l be $Z_1^G = \{Z_1, \ldots, Z_5\}$, $W_1^G = \{W_1, \ldots, W_5\}$, U^G, V^G of length 5, 5, 6, 10 respectively. Via a fixed isomorphism $G \xrightarrow{\cong} \mathrm{A}_5$ we identify the elements of G with those of A_5. We may suppose that $\tau = (23)(45)$, $\tau' = (24)(35)$, $G_V = \langle (12)(34), (345)\rangle$, $Z_1^\sigma = Z_{1^\sigma}$, $W_1^\sigma = W_{1^\sigma}$ for all $\sigma \in G$. Since $\langle \tau, \tau'\rangle$ fixes both Z_1, W_1 and no other point of l, τ' induces a homology on the Baer subplane Fix(τ). Interchanging W_1, Z_1 if necessary, we may suppose that $\langle \tau, \tau'\rangle$ fixes a line $l' \ne l$ through Z_1 and exactly 5 points of l' distinct from Z_1. Now $\langle (345)\rangle$ permutes these 5 points, fixing at least two, say X_1, Y_1. By assumption G_{X_1}, $G_{Y_1} \subsetneq G$ and so $G_{X_1} = G_{Y_1} = \langle \tau, \tau', (345)\rangle$. We may write $X_1^G = \{X_1, \ldots, X_5\}$ where $X_1^\sigma = X_{1^\sigma}$ for all $\sigma \in G$ and similarly for Y_1^G. The points $X_1X_2 \cap l$, $Y_1Y_2 \cap l$ are fixed by $\langle (12)(34), (345)\rangle$ and so $X_1X_2 \cap l = Y_1Y_2 \cap l = V$.

Since (345) fixes $X_2Y_1 \cap l$ we have $X_2Y_1 \cap l \in \{W_1, W_2, Z_1, Z_2, V\}$. If $X_2Y_1 \cap l = W_1$ then $X_3Y_1 \cap l = (X_2Y_1 \cap l)^\tau = W_1^\tau = W_1$ so that X_2, X_3, Y_1, W_1 are collinear, i.e. $W_1 = X_2X_3 \cap l = (X_1X_2 \cap l)^{(13)(45)} \in V^G$, a contradiction. Similarly $X_2Y_1 \cap l \neq W_2$. If $X_2Y_1 \cap l = Z_1$ then X_1, X_2, Y_1, Z_1 are collinear, violating $X_1X_2 \cap l = V$. Similarly $X_2Y_1 \cap l \neq Z_2$. If $X_2Y_1 \cap l = V$ then X_1, X_2, Y_1, V are collinear, violating $X_1Y_1 \cap l = Z_1$. This proves (12).

Now by (10), (11), (12) and the dual statements, G must fix at least a triangle of Π. At most one side of this triangle is of type $(10, \mathrm{i})$, since otherwise G fixes pointwise a Baer subplane, contrary to 2.2. Furthermore this triangle cannot have three sides of type $(10, \mathrm{ii})$, since $\mathrm{Fix}(E)$ consists of exactly $q+1$ collinear points and one additional point. Hence G has a fixed line l' of type $(10, \mathrm{i})$, and G fixes another point X outside l', which lies on $q+1$ lines of type $(10, \mathrm{ii})$. We may take l to be one of these $q+1$ lines of type $(10, \mathrm{ii})$.

Let $\rho \in G$ be an element of order p such that $\tau\rho\tau = \rho^{-1}$. Then $\mathrm{Fix}(\rho)$ is a Baer subplane of Π, on which τ induces a homology or acts trivially. Hence the number of fixed points of $\langle \rho, \tau \rangle$ on l is

$$\sum_{\nu=1}^{w} F_\nu(\rho, \tau) = 2 \text{ or } q+1.$$

Writing $|G_1| = |G_2| = |G| = \frac{1}{2}q(q^2-1)$, $|G_\nu| = qd_\nu$ for $\nu = 3, 4, \ldots, w$, equation (6) yields

$$\sum_{\substack{3 \le \nu \le w \\ d_\nu \text{ even}}} \frac{1}{d_\nu}\big(q-1\big) = q-1.$$

By (9) we have

$$F_\nu(\rho, \tau) = \begin{cases} (q-1)/2d_\nu, & d_\nu \text{ even}, \\ 0, & d_\nu \text{ odd}, \end{cases} \qquad \nu = 3, 4, \ldots, w,$$

and so

$$\sum_{\nu=1}^{w} F_\nu(\rho, \tau) = \tfrac{1}{2}(q+3),$$

a contradiction. This eliminates the case $q \equiv 5 \bmod 8$, and so we have proved that

(13) $\quad q \equiv 1 \bmod 8.$

Now G has a single conjugacy class of elements of order 4, from which we may choose a representative γ. We have

(14) $\quad \sum_{\nu=1}^{w} F_\nu(\gamma) = 2$ or $q+1$, or possibly $q^{1/2}+1$ if q is a square.

Type	G_ν
1	G
2	order qd, $d \mid \frac{1}{2}(q-1)$
3	order $p^{m-e}(p^e-1)$, $2e \mid m$, $q=p^m$. (There may be several conjugacy classes of such subgroups in G.)
4	dihedral of order $q-1$
5	dihedral of order $q+1$
6	cyclic of order $\frac{1}{2}(q+1)$
8a, 8b	PSL(2, $q^{1/2}$) (For q a square only. Such a G_ν is said to be of type 8a or 8b according as it is or is not conjugate to the standard embedding PSL(2, $q^{1/2}$) $\hookrightarrow G$.)
9a, 9b	PGL(2, $q^{1/2}$) (For q a square only. A type 9a, 9b subgroup is the normalizer in G of a subgroup of type 8a, 8b respectively.)
10a, 10b	S_4 (For $q=17$, 25, 41 only. A type 10a, 10b subgroup is the normalizer in G of a conjugate of E_1, E_2 respectively.)
11a, 11b	A_4 (For $q=17$ only. A type 11a, 11b subgroup contains a conjugate of E_1, E_2 respectively.)
12a, 12b	A_5 (For $q=9$, 41, 49, 89 only. A type 12a, 12b subgroup contains a conjugate of E_1, E_2 respectively.)

TABLE 3B [2]

However, G has *two* conjugacy classes of each of the following, between which we must distinguish: elements of order p, elementary abelian subgroups of order 4, and subgroups isomorphic to A_4, S_4, A_5 (for $q(q^2-1)\equiv 0$ mod 5), and PSL(2, $q^{1/2}$), PGL(2, $q^{1/2}$) (for q a square).

The two conjugacy classes of elementary abelian subgroups of order 4 are represented by

$$E_1 = \left\langle \tau, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle, \qquad E_2 = \left\langle \tau, \begin{pmatrix} 0 & \epsilon \\ -\epsilon^{-1} & 0 \end{pmatrix} \right\rangle$$

where $\tau = \operatorname{diag}(i,-i)$; i, ϵ are elements of GF(q) such that $i^2=-1$ and ϵ is a given non-square. The two conjugacy classes of elements of order p are represented by

$$\rho_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}.$$

By [3], [12] and Lemma 2.8, the only possibilities for G_ν satisfying (8) are as listed in Table 3B.

We compute the entries of Tables 3C,D using (9) and the values of $[G : \mathrm{N}_G(H)]$ obtained from [3] for those subgroups $H < G$ concerned. Only certain of the computation of these entries bear explicit mention here. For $q=9$, we see that E_1 is normalized by the element

$$\begin{pmatrix} 1-i & 1-i \\ -1-i & 1+i \end{pmatrix} = \begin{pmatrix} i & -1 \\ 1+i & -1 \end{pmatrix} \begin{pmatrix} 1 & -1+i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1-i & i \end{pmatrix}$$

[2] *Note in Table 3B.* Although type 10 also occurs for $q=9$, this has already been listed as type 9. Similar duplications have been likewise suppressed.

of order 3, so that the elements of order 3 in a type 12a subgroup are conjugate to ρ_2 in G. Thus if $q = 9$ and G_ν is of type 12a, (9) gives

$$F_\nu(\rho_1) = \frac{18}{60} \times 0 = 0, \qquad F_\nu(\rho_2) = \frac{18}{60} \times 10 = 3, \qquad F_\nu(\rho_2, \tau) = \frac{6}{60} \times 10 = 1,$$

and similarly for G_ν of type 12b.

If q is a square and G_ν is of type 9a, then all elements of order p in G are conjugate to ρ_1 in G. Such a G_ν contains $\frac{1}{24}q^{1/2}(q-1)$ conjugates of E_1 (respectively, E_2) in G, and $\frac{1}{8}q^{1/2}(q-1)$ conjugates of E_2 (resp., E_1) in G, if $q^{1/2} \equiv 1$ (resp., 3) mod 4. By (9) we obtain the entries of Tables 3C,D for G_ν of type 9. The calculation of the remaining entries is straightforward.

Extending our previous notation, we have $n_8 = n_{8\mathrm{a}} + n_{8\mathrm{b}}$, $n_9 = n_{9\mathrm{a}} + n_{9\mathrm{b}}$, etc. For orbits of type 2, we write $|G_\nu| = qd_\nu$, and introduce the following abbreviations:

$\Sigma_1 = \sum (q-1)/d_\nu$, (sum over all orbits of type 2);

Σ_2, $\Sigma_{2'}$, Σ_4, $\Sigma_{2,4'}$ (sum with the same summand $(q-1)/d_\nu$ but over $2 \mid d_\nu$, $2 \nmid d_\nu$, $4 \mid d_\nu$, $d_\nu \equiv 2 \bmod 4$, respectively).

By referring to Tables 3C,D and subtracting (14) from (6), we obtain:

(15) If γ fixes $q+1$ points of l, then the following must hold.

(i) $\Sigma_{2,4'} = n_4 = n_5 = n_9 = n_{10} = n_{11} = n_{12} = 0$.

(ii) If there is a type 3 orbit for some e, then $p^e \equiv 1 \bmod 4$.

(iii) If $q \equiv 9 \bmod 16$, then $n_8 = 0$.

Type	$[G:G_\nu]$	$F_\nu(\tau)$	$F_\nu(E_1)$	$F_\nu(E_2)$
1	1	1	1	1
2	$(q^2-1)/2d$	$\begin{cases}(q-1)/d, & 2 \mid d \\ 0, & 2 \nmid d\end{cases}$	0	0
3	$\dfrac{p^e(q^2-1)}{2(p^e-1)}$	$\dfrac{q-1}{p^e-1}$	0	0
4	$\frac{1}{2}q(q+1)$	$\frac{1}{2}(q+1)$	3	3
5	$\frac{1}{2}q(q-1)$	$\frac{1}{2}(q-1)$	0	0
6	$q(q-1)$	0	0	0
8a	$q^{1/2}(q+1)$	$q^{1/2}\pm 1$	1 ± 1	1 ∓ 1
8b	$q^{1/2}(q+1)$	$q^{1/2}\pm 1$	1 ∓ 1	1 ± 1
9a	$\frac{1}{2}q^{1/2}(q+1)$	$q^{1/2}$	2 ± 1	2 ∓ 1
9b	$\frac{1}{2}q^{1/2}(q+1)$	$q^{1/2}$	2 ∓ 1	2 ± 1
10a	$\frac{1}{48}q(q^2-1)$	$\frac{3}{8}(q-1)$	$\begin{cases}4, & q\equiv 1 \bmod 16 \\ 1, & q\equiv 9 \bmod 16\end{cases}$	$\begin{cases}4, & q\equiv 1 \bmod 16 \\ 3, & q\equiv 9 \bmod 16\end{cases}$
10b	$\frac{1}{48}q(q^2-1)$	$\frac{3}{8}(q-1)$	$\begin{cases}4, & q\equiv 1 \bmod 16 \\ 3, & q\equiv 9 \bmod 16\end{cases}$	$\begin{cases}4, & q\equiv 1 \bmod 16 \\ 1, & q\equiv 9 \bmod 16\end{cases}$
11a	204	$\frac{1}{4}(q-1)=4$	2	0
11b	204	$\frac{1}{4}(q-1)=4$	0	2
12a	$\frac{1}{120}q(q^2-1)$	$\frac{1}{4}(q-1)$	2	0
12b	$\frac{1}{120}q(q^2-1)$	$\frac{1}{4}(q-1)$	0	2

TABLE 3C [3]

[3] *Note in Tables 3C,D.* For $\pm$, $\mp$ read the *upper* sign if $q^{1/2}\equiv 1 \bmod 4$; the *lower* sign if $q^{1/2}\equiv 3 \bmod 4$ (for q a square only). This convention is followed throughout §3. The entries k_1, k_2 in Table 3D are non-negative integers such that $k_1+k_2=(p^{m-e}-1)/(p^e-1)$; the values of k_1, k_2 depend not only on e but on the particular conjugacy class of the type 3 subgroup G_ν.

Type	$F_\nu(\rho_1)$	$F_\nu(\rho_2)$	$F_\nu(\rho_1,\tau)$	$F_\nu(\rho_2,\tau)$	$F_\nu(\gamma)$
1	1	1	1	1	1
2	$\frac{q-1}{2d}$	$\frac{q-1}{2d}$	$\begin{cases} \frac{q-1}{2d}, & 2 \mid d \\ 0, & 2 \nmid d \end{cases}$	$\begin{cases} \frac{q-1}{2d}, & 2 \mid d \\ 0, & 2 \nmid d \end{cases}$	$\begin{cases} \frac{q-1}{d}, & 4 \mid d \\ 0, & 4 \nmid d \end{cases}$
3	$k_1 p^e$	$k_2 p^e$	k_1	k_2	$\begin{cases} \frac{q-1}{p^e-1}, & p^e \equiv 1 \bmod 4 \\ 0, & p^e \equiv 3 \bmod 4 \end{cases}$
4	0	0	0	0	1
5	0	0	0	0	0
6	0	0	0	0	0
8a	$2q^{1/2}$	0	1 ± 1	0	$\begin{cases} q^{1/2} \pm 1, & q \equiv 1 \bmod 16 \\ 0, & q \equiv 9 \bmod 16 \end{cases}$
8b	0	$2q^{1/2}$	0	1 ± 1	$\begin{cases} q^{1/2} \pm 1, & q \equiv 1 \bmod 16 \\ 0, & q \equiv 9 \bmod 16 \end{cases}$
9a	$q^{1/2}$	0	1	0	$\frac{1}{2}(q^{1/2} \pm 1)$
9b	0	$q^{1/2}$	0	1	$\frac{1}{2}(q^{1/2} \pm 1)$
10a	0	0	0	0	$\frac{1}{8}(q-1)$
10b	0	0	0	0	$\frac{1}{8}(q-1)$
11a	0	0	0	0	0
11b	0	0	0	0	0
12a	0	$\begin{cases} 3, & q = 9 \\ 0, & \text{else} \end{cases}$	0	$\begin{cases} 1, & q = 9 \\ 0, & \text{else} \end{cases}$	0
12b	$\begin{cases} 3, & q = 9 \\ 0, & \text{else} \end{cases}$	0	$\begin{cases} 1, & q = 9 \\ 0, & \text{else} \end{cases}$	0	0

Table 3D

Type	1	2	2	2	3	4	5	8a	8b	9a	9b	12a	12b
$\lvert G_\nu \rvert$	360	9	18	36	6	8	10	12	12	24	24	60	60
(ix)	1					1	1						
(x)	1						1	1			1		
(xi)	1						1		1	1			
(xii)	1				1					1		1	
(xiii)	1				1						1		1
(xiv)	1	1	1							1		1	
(xv)	1	1	1								1		1

TABLE 3E [4]

We next prove the following:

(16) The orbit structure of G on the points of l belongs to one of the following thirteen cases:

(i) $q+1$ fixed points and one orbit of type 6;

(ii) 2 fixed points and the remaining orbits of type 2 with $\Sigma_2 = \Sigma_{2'} = q-1$;

(v) 2 fixed points, one point orbit of type 3 for some e, and the remaining orbits of type 2 with

$$\Sigma_2 = (p^e - 2)\frac{q-1}{p^e-1}, \qquad \Sigma_{2'} = 0;$$

(vi) 2 fixed points, $\frac{1}{2}(q^{1/2}-1)$ orbits of each of the types 8a and 8b, and the remaining orbits of type 2 with $\Sigma_2 = (1 \mp 1)(q^{1/2}-1)$, $\Sigma_{2'} = (1 \pm 1)(q^{1/2}-1)$, where q is a square, $q^{1/2} \equiv \pm 1 \bmod 4$;

(vii) 2 fixed points, $\frac{1}{2}(q^{1/2}-1)$ orbits of type 8a, and the remaining orbits of type 2 with $\Sigma_2 = \frac{1}{2}(q-1)$, $\Sigma_{2'} = \frac{1}{2}(q^{1/2}-1)(q^{1/2}+3)$, where q is a square with $q^{1/2} \equiv 1 \bmod 4$;

(viii) as in (vii) with 8b in place of 8a;

(ix)–(xv) as per Table 3E, where $q = 9$ in each case.

In proving (16), we may assume (by the remarks following (10)) that $n_1 \leq q$, and that

(17) $\sum_{\nu=1}^{w} F_\nu(E_i) = 2$, or possibly $q^{1/2}+1$ (if q is a square), $i = 1, 2$;

(18) $n_6 = 0$.

We shall first suppose that an orbit of type 10, 11 or 12 occurs.

If $q = 17$, then (17) gives $n_4 = n_{10} = 0$, $n_1 + 2n_{11a} = n_1 + 2n_{11b} = 2$. We *cannot* have $n_{11a} = n_{11b} = 1$, by (5). Hence $n_{11} = 0$.

[4] Thus by Table 3E, (l) contains one fixed point and one point orbit of each of the types 4, 5 in case (ix), etc.

If $q=41$, then (17) gives $n_4=n_{10}=0$, $n_1+2n_{12\mathrm{a}}=n_1+2n_{12\mathrm{b}}=2$. Since orbits of type 2, 4, 10, 12 have lengths divisible by 7, (5) gives $n_1+n_5\equiv 2 \bmod 7$. Supposing that $n_1=0$, $n_{12\mathrm{a}}=n_{12\mathrm{b}}=1$, then (5) obtains $n_5=0$, a contradiction. Hence $n_{12}=0$.

If $q=49$, then orbits of type 3, 4, 8, 9 have lengths divisible by 25. Therefore (5) gives $n_1+n_5+5n_{12}\equiv 2$ mod 25, $1176n_5+980n_{12}\le 2402$, while (17) requires $n_1\le 8$. This implies $n_{12}=0$.

If $q=89$, then either $n_{12}=0$, or (17) gives $n_{12\mathrm{a}}=n_{12\mathrm{b}}=1$, $n_1=n_4=0$. But the latter violates (5), so we must have $n_{12}=0$.

If $q=25$, then (5) gives $n_1+n_5\equiv 2 \bmod 13$, while (17) implies $n_1+3n_4\le 6$. Hence $n_1+n_5=2$. Supposing $n_{10}>0$, then (5) gives $n_{10}=1$, $n_5\le 1$. But we *cannot* have $n_1=n_5=1$ by (6), so we must have $n_1=2$, $n_5=0$. Then (17) implies $n_4=1$, which contradicts (14), (15). Hence $n_{10}=0$.

If $q=9$ then $G\cong \mathrm{A}_6$, and the only solutions of (5), (6), (14), (17) having $n_{12}>0$ are given by cases (xii) through (xv) of (16), plus an additional seven solutions in which γ fixes precisely 2 points X, Y of l, both in a single orbit of length 10. We may assume that we have one of the latter solutions, and that $|G_1|=36$, G_1 of type 2. Now γ induces a homology on the Baer subplane $\mathrm{Fix}(\gamma^2)$, and so γ fixes exactly 10 lines through one of X, Y and exactly 2 lines through the other. Choose $g\in G$ such that $X^g=Y$. Then $g^{-1}\gamma g$ fixes as many lines through Y as γ does through X. But by Sylow's Theorem, $\langle g^{-1}\gamma g\rangle$, $\langle\gamma\rangle$ are conjugate in G_Y and must fix an equal number of lines through Y. This is a contradiction, and so for each q, we may assume that

(19) $\quad n_{10}=n_{11}=n_{12}=0.$

From (5) we have $n_4\le 1$. Suppose that $n_4=1$. We may assume that $n_5=0$ (for otherwise (5) gives $n_5=n_1=1$, $w=3$ so (17) yields $q=9$, but this is case (16, ix)). Now (5) gives $n_3=0$, $n_1\equiv 2$ mod $\frac{1}{2}(q+1)$ while (17) requires $n_1\le q^{1/2}-2$; hence $n_1=2$. By (15), γ fixes exactly $q^{1/2}+1$ points of l (q being necessarily a square), so (14) yields $\frac{1}{2}(q^{1/2}\pm 1)n_9\le q^{1/2}-2$. But (5) implies that n_9 is odd, so we must have $n_9=1$. Then (17) gives $n_8=q^{1/2}-6$ and in particular $q\ge 49$. Now (6) yields $q=49$ or 81, both of which are eliminated by (14), (15). Hence we may assume that

(20) $\quad n_4=0.$

By (5), (17) we have $n_5\le 1$. Supposing that $n_5=1$, then (5) implies $n_1\equiv 1 \bmod \frac{1}{2}(q+1)$ while (17) requires $n_1\le q^{1/2}+1$; hence $n_1=1$. Since an orbit of type 3 has length at least $\frac{1}{2}(q^2+1)(q^{1/2}+1)$, we must have $n_3=0$. Now (17) gives $n_8+2n_9=q^{1/2}$, while (6) implies $\Sigma_2+(q^{1/2}\pm 1)n_8+q^{1/2}n_9=\frac{1}{2}(q+1)$. Hence $2\Sigma_2+(q^{1/2}\pm 2)n_8=1$, and so $\Sigma_2=0$, $q=9$, $n_8=n_9=1$ (where either $n_{8\mathrm{a}}=n_{9\mathrm{b}}=1$ or $n_{8\mathrm{b}}=n_{9\mathrm{a}}=1$). Then (5) gives $n_2=0$, and so we have case (x) or (xi) of (16). Hence we may assume that

(21) $\quad n_5=0.$

Now (5), (17) yield $n_1 \equiv 2 \bmod \frac{1}{2}(q+1)$, $n_1 \le q^{1/2}+1$ and so

(22) $n_1 = 2$.

By (5) we have $n_3 \le 1$; suppose that $n_3 = 1$. Then (5), (6) yield $\Sigma_{2'} + (q^{1/2} \mp 1)n_8 = 0$, i.e. $\Sigma_{2'} = n_8 = 0$. By (17) we have $n_9 = 0$ or $\frac{1}{2}(q^{1/2}-1)$. But $n_9 \ne \frac{1}{2}(q^{1/2}-1)$ (for otherwise (15) requires that γ fix exactly $q^{1/2}+1$ points of l, so $2 + \frac{1}{2}(q^{1/2} \pm 1)n_9 \le q^{1/2}+1$, a contradiction). Hence $n_9 = 0$ and (5) yields case $(16, \mathrm{v})$. Otherwise we may assume that

(23) $n_3 = 0$.

If $n_8 = n_9 = 0$ then (5), (6), (17) obtain case $(16, \mathrm{ii})$. Hence we may assume that

(24) n_8, n_9 are not both 0.

If $n_9 \ne 0$ then (14), (15) give $n_9 = 2$, $\Sigma_4 = 0$ and $q^{1/2} \equiv 3 \bmod 4$; but then (17) yields $n_8 = q^{1/2} - 5$ and (15) gives $\Sigma_2 = 4q^{1/2} - 6 \equiv 2 \bmod 4$, contradicting $\Sigma_2 = \Sigma_{2,4'} \equiv 0 \bmod 4$. Hence $n_9 = 0$.

Now (17), (24) give $n_8 = \frac{1}{2}(q^{1/2}-1)$ or $q^{1/2}-1$. If $n_8 = q^{1/2}-1$ then (5), (6) give case $(16, \mathrm{vi})$. Otherwise $n_8 = \frac{1}{2}(q^{1/2}-1)$ and (5), (6) yield $\Sigma_2 = \frac{1}{2}(q^{1/2}-1)(q^{1/2}+2 \mp 1)$, $\Sigma_{2'} = \frac{1}{2}(q^{1/2}-1)(q^{1/2}+2 \pm 1)$. Since the highest power of 2 dividing $q-1$ must also divide $\Sigma_{2'}$, we obtain $q \equiv 1 \bmod 4$, giving case (vii) or (viii) of (16). This completes the proof of (16).

Let us now suppose that the action of G on (l) is given by one of the cases (ix)–(xv) of (16), so that $q = 9$ and G fixes a point X of l. Also E_1, γ fix 4, 2 points of l respectively. We claim that this leads to case (iv) of 1.2. Accordingly we suppose that G fixes a further line l_2 through X, and derive a contradiction. We may apply (16) to the set of points of l_2, and dually to the set of lines through X. Since E_1 must fix 4 points of l_2, we see that l_2 belongs to one of the types (vi), (ix)–(xv); hence γ fixes exactly 2 points of l_2. Similarly γ fixes exactly 2 lines through X. Hence $\mathrm{Fix}(\gamma)$ is just a triangle, which is the required contradiction. We may therefore assume that

(25) types (ix) through (xv) of (16) do not occur.

Therefore by (16), (25) and the dual statements, G fixes at least a triangle having (say) $l_1{=}l$, l_2, l_3 as its sides. In particular, $\mathrm{Fix}(\rho_1)$ is a subplane, on which τ either acts trivially or induces a homology or Baer collineation.

If l is of type $(16, \mathrm{ii})$ then ρ_1 fixes $q+1$ points of l, so that $\mathrm{Fix}(\rho_1)$ is a Baer subplane of Π; but the number of points of l fixed by $\langle \rho_1, \tau \rangle$ is then $\frac{1}{2}(q+3) \notin \{2,\, q^{1/2}+1,\, q+1\}$, a contradiction.

If l is of type $(16, \mathrm{v})$ then ρ_i, $\langle \rho_i, \tau \rangle$ fix $M + k_i p^e + 1$, $M + k_i + 1$ points of l respectively, for $i = 1, 2$, where

$$M = 1 + \tfrac{1}{2}(p^e - 2)\frac{q-1}{p^e-1}, \qquad k_1 + k_2 = \frac{p^{m-e}-1}{p^e-1};$$

Type	Number of points of l fixed by E_1	E_2
(i)	$q+1$	$q+1$
(vi)	$q^{1/2}+1$	$q^{1/2}+1$
(vii)	$q^{1/2}+1$	2
(viii)	2	$q^{1/2}+1$

Table 3F

we may choose i such that $k_i > 0$ and hence $(M+k_i)^2 \leq M+k_ip^e$ by Bruck's Theorem [11,p.81], contradicting the easily verified relation $M^2 > M + (p^{m-e}-1)p^e/(p^e-1)$.

We tabulate in 3F the number of points of l fixed by E_1, E_2 for each of the remaining possible types for l.

If l is of type $(16, \mathrm{vii})$ then E_1 (respectively, E_2) induces a Baer collineation (resp., a homology) on the Baer subplane Fix(τ). Hence either l_2 or l_3 has exactly $q^{1/2}+1$ points fixed by E_1 and $q+1$ points fixed by E_2, which is impossible by Table 3F. Hence type $(16, \mathrm{vii})$, and similarly type $(16, \mathrm{iii})$, do not occur.

If l is of type $(16, \mathrm{vi})$ then E_1, E_2 induce Baer collineations of Fix(τ), so l_2, l_3 are also of type $(16, \mathrm{vi})$. We may take G_1 to be a subgroup of type 8a, $N = \mathrm{N}_G(G_1) \cong \mathrm{PGL}(2, q^{1/2})$ and $\alpha \in N \setminus G_1$ an involution. Then G_1 fixes $q^{1/2}+1$ points of each of l, l_2, l_3 and so Fix(G_1) is a subplane of order $q^{1/2}$, on which α must act trivially or induce either a Baer collineation or a homology. However, $\mathrm{Fix}(G_1, \alpha) = \mathrm{Fix}(N)$ consists of precisely the triangle with sides l, l_2, l_3, a contradiction.

Otherwise l, l_2, l_3 are of type $(16, \mathrm{i})$ and so Fix(G) is a Baer subplane of Π. But a subgroup of type 6 fixes more than $q+1$ points of l, a contradiction. □

4. Proof of Theorem 1.1

If $n \leq 8$ then Π is Desarguesian by 2.7, and the conclusion of the theorem is easily verified by [19] for n odd, [7] for n even. Hence we may assume that

(26) $n \geq 9$, $q \geq 11$ and G is simple.

Now suppose that G fixes a line l of Π. If G acts nontrivially on (l) then by [12] we must have $q = 11$, $n = 10$ and G acts transitively on the points of l. But then each involution in G fixes exactly 3 points of l, a contradiction. Otherwise G fixes l pointwise. If G consists of elations of Π then these elations have a common centre (since the conjugates of a given $g \in G \setminus 1$ have a common centre, and generate G) and so $|G| = \frac{1}{2}q(q^2-1) \leq n < q$, which is absurd. Therefore G contains a homology. By the simplicity of G, we see that G contains no nontrivial elations. Therefore all elements of G are homologies with axis l which (by

[11,Theorems 4.13,4.25]) must have a common centre, and so $|G| \le n-1$, a contradiction. Hence we may assume that

(27) G acts irreducibly on Π.

Let P be a Sylow p-subgroup of G. By Theorem 2.3 we have

(28) Fix(P) is not a subplane of Π.

Next we show that

(29) G has no orbit of length $q+1$.

For suppose that $|X^G| = q+1$ for some point X of Π. Since $n < q$, the points of X^G are not collinear, and so 2.9(ii) demands that X^G be a $(q+1)$-arc. Thus $n = q-1$ is even, q is odd, and X^G is a hyperoval. Let $\tau \ne \tau'$ be commuting involutions in G. Then τ is a Baer collineation of Π (not an elation; otherwise by 2.4(ii) and duality, we may suppose that τ, τ' are elations with a common axis l, so that l is fixed by $\langle \mathrm{C}_G(\tau), \mathrm{C}_G(\tau')\rangle = G$, contrary to (27)). Thus $n = q-1$ is a square, and in particular $q \equiv 1 \bmod 4$ and q is not a square.

Clearly P fixes an antiflag (X_1, l), and by (28) any further fixed points of P lie on l and their number is divisible by p. In particular Fix(P) is not a triangle and $N = \mathrm{N}_G(P)$ fixes l. Now $|l^G| = q+1$ and the lines of l^G are not concurrent by (27), and so by 2.9(ii), l^G is a dual hyperoval of Π.

Now P acts regularly on $l^G \setminus \{l\}$ and hence also on (l) (since each point of (l) meets a unique line of $l^G \setminus \{l\}$) so that P fixes only the antiflag (X_1, l) and we must have $X_1 = X$. The point set $\Omega = \bigcup_{g\in G}(l^g)$ is a G-orbit of length $\frac{1}{2}q(q+1)$ since G acts 2-transitively on l^G. Because $|X^G| + |\Omega| < q^2 - q + 1$ we may choose a point Y of Π outside $X^G \cup \Omega$. Now $[G\colon G_Y] = |Y^G| \le (q^2-q+1) - (q+1) - \frac{1}{2}q(q+1) = \frac{1}{2}q(q-5)$ and so $|G_Y| \ge q+6$. Using this, the fact that q is not a square, and the fact that no conjugate of P fixes Y, together with [12] and Lemma 2.8, we have $G_Y \cong \mathrm{A}_5$ and $q \in \{19, 31, 49\}$; but in none of these cases is $q-1$ a square. By contradiction we have verified (29). Next we show that

(30) q is odd.

For suppose that q is even. Then G has a single conjugacy class of involutions. If G contains Baer involutions then counting in two different ways the number of pairs (X, g) such that X is a point of Π and $g \in P$ fixes X, we have

$$qw = (n^2+n+1) + (q-1)(n + n^{1/2} + 1)$$

where w is the number of point orbits of P on Π (see [21]), but for q even this yields $q \,\big|\, n - n^{1/2}$, contradicting $n < q$. Otherwise $P \setminus 1$ consists of perspectivities of Π, and these do not have a common axis or centre (for a common axis or centre would be fixed by $\mathrm{N}_G(P)$, contrary to (27), (29)). By 2.4, $P \setminus 1$ consists of homologies fixing a common triangle $X_0X_1X_2$, and again by (29), $\mathrm{N}_G(P)$ permutes X_0, X_1, X_2 transitively. If $P(X_0, X_1X_2)$ is the subgroup of P consisting of (X_0, X_1X_2)-homologies, then $|P(X_0, X_1X_2)| = 1+\frac{1}{3}|P\backslash 1|$. But by (26), q and $1+\frac{1}{3}(q-1)$ cannot both be powers of 2, a contradiction, which proves (30).

(31) $\quad \mathrm{Fix}(P) = \emptyset$.

For suppose that Fix(P) is nonempty. Then $N = \mathrm{N}_G(P)$ acts on Fix(P) without fixing any point or line, by (29). Therefore by 2.1, Fix(P) is a triangle with sides l_0, l_1, l_2 (say) on which N induces all three cyclic permutations. This triangle is fixed elementwise by a subgroup $H < N$ with $|H| = \frac{1}{6}q(q-1)$. Since N is the unique maximal subgroup of G containing H, we have $G_{l_0} = H$ and $|l_0^G| = 3(q+1)$.

Let $\tilde{P} \neq P$ be another Sylow p-subgroup of G, and let l be a line of Π fixed by $\tilde{P}$. We have $P_l = 1$ (for otherwise l is fixed by $\langle \tilde{P}, P_l\rangle = G$, contradicting (27)) and so $|l^P| = q$. Now l does not pass through any vertex of the triangle Fix(P) (for otherwise $|l^P| \leq n-1 < q$, a contradiction). Therefore we have three distinct points $X_i = l_i \cap l$, $i = 0, 1, 2$. We have a relationship between stabilizers of X_0 given by $H_{X_0} = CP_{X_0}$ where $C = C(X_0)$ is cyclic, $C \cap P_{X_0} = 1$. Now $[P\colon P_{X_0}] \leq n-1 < q$ implies $P_{X_0} \neq 1$, and C acts semiregularly on $P_{X_0} \setminus 1$ by conjugation so that $|C| \leq |P_{X_0}| - 1$. Also $[H\colon H_{X_0}] \leq n-1 < q-1$ and so

$$|P_{X_0}|^2 > |C||P_{X_0}| = |H_{X_0}| > q/6.$$

Since $|P_{X_0}|$ is a power of p and $q \equiv 1 \bmod 3$, it follows that $|P_{X_0}| \geq q^{1/2}$. Similarly $|P_{X_1}|$, $|P_{X_2}| \geq q^{1/2}$ and so $|X_i^P| \leq q^{1/2}$, $i = 0, 1, 2$. Now $|l^P| \leq |X_0^P||X_1^P| \leq q^{1/2}q^{1/2} = q$. Therefore equality holds and $|X_i^P| = q^{1/2}$, $i = 0$, 1, 2. In particular, q is a square.

If $\sigma \in G$ satisfies $l_0^\sigma = l_1$ then $\sigma \in N$ and in particular $\sigma \notin \tilde{P}$. Therefore we have a partition

$$l^G = l^{\tilde{N}} \cup l_0^{\tilde{P}} \cup l_1^{\tilde{P}} \cup l_2^{\tilde{P}}$$

where $\tilde{N} = \mathrm{N}_G(\tilde{P})$, $|l^{\tilde{N}}| = 3$, $|l_i^{\tilde{P}}| = q$, $i = 0, 1, 2$. We shall consider the incidence structure formed by the lines of l^G and the set Ω of points of the form $a \cap a'$ where the lines a, a' are fixed by distinct Sylow p-subgroups of G. Since $X_i^{\tilde{P}}$ consists of $q^{1/2}$ points of l, we must have one of the following three cases:

(i) $X_0^{\tilde{P}} = X_1^{\tilde{P}} = X_2^{\tilde{P}}$;

(ii) two of the $X_i^{\tilde{P}}$ are equal, $i = 0, 1, 2$: say $X_0^{\tilde{P}} = X_1^{\tilde{P}}$, $X_0^{\tilde{P}} \cap X_2^{\tilde{P}} = \emptyset$; or

(iii) $X_0^{\tilde{P}}$, $X_1^{\tilde{P}}$, $X_2^{\tilde{P}}$ are mutually disjoint.

In case (i), if $l' \neq l$ is another side of the triangle Fix$(\tilde{P})$ then $Y_0^{\tilde{P}} = Y_1^{\tilde{P}} = Y_2^{\tilde{P}}$ (as is seen by applying the appropriate element of $N \cap \tilde{N}$) where $Y_i = l_i \cap l'$, $i = 0, 1, 2$. But then $l_i^{\tilde{P}}$ consists of all lines meeting both $X_i^{\tilde{P}}$ and $Y_i^{\tilde{P}}$, $i = 0, 1, 2$ so that $l_0^{\tilde{P}} = l_1^{\tilde{P}}$, a contradiction.

In case (ii) we obtain the partition $\Omega = X_0^G \cup X_2^G$ where each point of X_0^G (respectively, X_2^G) lies on exactly $2q^{1/2}+1$ (resp., $q^{1/2}+1$) lines of l^G, and each line of l^G contains $q^{1/2}$ points of each of X_0^G, X_2^G.

Counting in two different ways the number of pairs of lines $\{a,\, a'\}$ where $a,\, a'$ are fixed by distinct Sylow p-subgroups of G, gives

$$\tfrac{1}{2}(3q+3)\cdot 3q = \binom{2q^{1/2}+1}{2}|X_0^G| + \binom{q^{1/2}+1}{2}|X_2^G|,$$

while counting the number of flags (X, a) such that $X \in \Omega,\ a \in l^G$ gives

$$2q^{1/2}\cdot 3(q+1) = (2q^{1/2}+1)|X_0^G| + (q^{1/2}+1)|X_2^G|.$$

This gives $3(q+1)q^{1/2} = (2q^{1/2}+1)|X_0^G| = (q^{1/2}+1)|X_2^G|$, which has no simultaneous solution in integers $|X_0^G|$, $|X_2^G|$ unless $q = 4$, which violates (26).

Therefore case (iii) must hold, and each point of Ω lies on $q^{1/2}+1$ lines of l^G. Counting flags (X, a) such that $X \in \Omega,\ a \in l^G$ gives

$$9q^{1/2}(q+1) = (q^{1/2}+1)|\Omega|.$$

The only solutions of this equation in integers satisfying (30) are $|\Omega| = 195$, 2465 for $q = 5^2$, 17^2 respectively.

We consider first the case $q = 17^2$, $|\Omega| = 2465$. Since $|P_{X_0}| = |\tilde{P}_{X_0}| = 17$, we have $G_{X_0} \cong \mathrm{PSL}(2, 17)$ or PGL(2,17). But $X_0^G \subseteq \Omega$, so that $X_0^G = \Omega$ and $G_{X_0} \cong \mathrm{PGL}(2, 17)$. Let $\gamma \in H$ be an element of order 4. Let w be the number of orbits of H on (l_0), and let $H_1{=}H,\ H_2{=}H,\ H_3,\ \ldots,\ H_w$ be the stabilizers (in H) of representatives from these respective point orbits. For $\nu = 3,\, 4,\, \ldots,\, w$ we have $H_\nu \not\supseteq P$, $[H\!:\!H_\nu] = 17k_\nu$ for some $k_\nu \bigm| 48$, and $\gamma,\ \gamma^2$ each fix exactly k_ν points in the ν-th such point orbit, by (9). Now l_0 has $n+1 = 2 + 17(k_3 + k_4 + \cdots + k_w)$ points, of which exactly $2 + k_3 + k_4 + \cdots + k_w$ are fixed by the involution γ^2. Therefore γ^2 is a Baer collineation and in particular, n is a square. Since $n \le q-1 = 288$ and $n \equiv 1$ mod 17, we have $n = 256$. Now γ fixes $2 + k_3 + k_4 + \cdots + k_w = 17$ points of l_0, and for the same reason γ fixes 17 points of l_1; thus $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\gamma^2)$. However $\gamma,\ \gamma^2$ fix exactly 9, 17 points of Ω, respectively (see Table 3D). This is a contradiction.

Otherwise we have $q = 25$, $|\Omega| = 195$. By the same method as above, we obtain $G_{X_0} \cong \mathrm{PSL}(2, 5)$ or PGL(2,5) (i.e. $|X_0^G| = 130$ or 65, respectively), $n = 16$, and if $\gamma \in H$ is an element of order 4 then $\mathrm{Fix}(\gamma) = \mathrm{Fix}(\gamma^2)$ is a Baer subplane. Since Ω consists of either three orbits each of length 65 or two orbits having respective lengths 65, 130, we may assume that $|X_0^G| = 65$, $G_{X_0} \cong \mathrm{PGL}(2, 5)$. Then $\gamma,\ \gamma^2$ fix exactly 0, 6 points of X_0^G respectively, a contradiction. This at last proves (31), and in particular

(32) $\qquad n^2 + n + 1 \equiv 0 \bmod p.$

If $n \equiv 1 \bmod p$ then (32) yields $p = 3$, whereas if $n \not\equiv 1 \bmod p$ then (32) implies that GF(p) contains a nontrivial cube root of 1. Therefore

(33) $\qquad$ either $p = 3$ or $p \equiv 1 \bmod 3$.

Let w be the number of orbits of G on the points of Π, and let $G_1,\ G_2, \ldots, G_w$ be the respective stabilizers of point representatives from these orbits. Then

(34) $\sum_{\nu=1}^{w}[G\!:\!G_\nu] = n^2+n+1 \le q^2-q+1.$

In particular, since $|G| = \frac{1}{2}q(q^2-1)$ we have

(35) $|G_\nu| \ge \frac{1}{2}(q+1), \qquad \nu = 1, 2, \ldots, w.$

By [3], [12] and Lemma 2.8 (cf. Tables 3A,B) the only possibilities for G_ν satisfying (26), (27), (31), (33), (35) are as listed in Table 4A.

Type	G_ν	$[G\!:\!G_\nu]$
3	order $p^{m-e}(p^e-1)$, $2e \mid m$, $q=p^m$	$\dfrac{p^e(q^2-1)}{2(p^e-1)}$
4	dihedral of order $q-1$	$\frac{1}{2}q(q+1)$
5	dihedral of order $q+1$	$\frac{1}{2}q(q-1)$
6	cyclic of order $\frac{1}{2}(q+1)$	$q(q-1)$
7	dihedral of order $\frac{1}{2}(q+1)$ (for $q\equiv 3 \bmod 4$ only)	$q(q-1)$
8	PSL$(2,q^{1/2})$ (for q a square only)	$q^{1/2}(q+1)$
9	PGL$(2,q^{1/2})$ (for q a square only)	$\frac{1}{2}q^{1/2}(q+1)$
10	S_4 (for $q=31$ only)	$\frac{1}{48}q(q^2-1)=620$
11	A_4 (for $q=13$, 19 only)	$\frac{1}{24}q(q^2-1)$
12	A_5 (for $q=19$, 31, 49, 61, 79, 81, 109 only)	$\frac{1}{120}q(q^2-1)$

Table 4A[5]

If G_1 is of type 6 or 7 then (34) gives $n=q-1$, $w=2$, $[G\!:\!G_2]=1$, contrary to (27). Therefore we must have

(36) $n_6 = n_7 = 0,$

where n_ν is the number of point orbits of type ν. We shall use the notations $F_\nu(\tau)$, ρ_1, $\pm$ in the same sense as in the proof of 1.2, and shall use (9) without explicit mention.

Suppose that G_1 is of type 3 for some e. Since

$$[G\!:\!G_1] = \frac{p^e(q^2-1)}{2(p^e-1)} \ge \tfrac{1}{2}(q+1)(q+q^{1/2}),$$

none of the remaining orbits are of type 3, 4 or 5. Since q is a square, we see that G_ν is of type 8, 9 or 12 (the latter type for $q=49$, 81 only) and $[G\!:\!G_\nu] \ge \frac{1}{2}(q+1)F_\nu(\tau)$, $\nu = 2, 3, \ldots, w$ (see Table 3C). The number of points of Π fixed by τ is

$$\frac{q-1}{p^e-1} + \sum_{\nu=2}^{w} F_\nu(\tau) \ge n+1,$$

[5] *Note in Table 4A.* We have suppressed duplicate entries (cf. Table 3B footnote).

while the total number of points in Π is

$$n^2+n+1=\frac{p^e(q^2-1)}{2(p^e-1)}+\sum_{\nu=2}^{w}[G\!:\!G_\nu],$$

from which

$$\begin{aligned}\frac{2}{q+1}\big(n^2+n+1\big)&=\frac{p^e(q-1)}{p^e-1}+\frac{2}{q+1}\sum_{\nu=2}^{w}[G\!:\!G_\nu]\\&\ge\frac{p^e(q-1)}{p^e-1}+\sum_{\nu=2}^{w}F_\nu(\tau)\ge n+q,\end{aligned}$$

contradicting $n\le q-1$. Therefore

(37) $\quad n_3=0.$

If $q=81$ then every point orbit has length divisible by 9, whereas n^2+n+1 is not divisible by 9 for any n, a contradiction. Hence

(38) $\quad q\ne 81.$

Suppose that n_8, n_9 are not both zero. Then q is a square, and the number of points of Π fixed by ρ_1 is $q^{1/2}(2n_{8\mathrm{a}}+n_{9\mathrm{a}})$, which we may assume to be positive (note that ρ_1 fixes no points in orbits of type 10, 11 or 12 due to (38)). Since $q>9$ this shows that Fix(ρ_1) is not a triangle. But P acts on Fix(ρ_1) without fixing any point or line by (31), and so by 2.1, Fix(ρ_1) is a subplane of order k, say. Now $\langle\rho_1,\tau\rangle$ fixes $(1\pm1)n_{8\mathrm{a}}+n_{9\mathrm{a}}$ points, and so $2n_{8\mathrm{a}}+n_{9\mathrm{a}}\ge k+1$. The total number of points fixed by ρ_1 is

$$k^2+k+1=q^{1/2}(2n_{8\mathrm{a}}+n_{9\mathrm{a}})\ge q^{1/2}(k+1),$$

and so $k^2>q-1$. This contradicts $k^2\le n\le q-1$, and so we obtain

(39) $\quad n_8=n_9=0.$

All remaining orbits are of types 4, 5, 10, 11 or 12, and every point orbit has length divisible by q.

We now eliminate the possibility of orbits of type 10, 11, 12. If $q=13$ and G_1 is of type 11, then $9\le n<q$, $n^2+n+1\equiv 0 \bmod q$ implies $n=9$ and $[G\!:\!G_1]=91$ so that $w=1$; but then τ fixes exactly $\frac14(q-1)=3$ points of Π, which is absurd.

q	$[G\colon C_G(\tau)]$	n	n^2+n+1
19	190	7, 11	57, 133
31	496	5, 25	31, 651
49	980	30, 32	931, 1057
61	1830	13, 47	183, 2257
79	3160	23, 55	553, 3081
109	5886	45, 63	2071, 4033

Table 4B

If $q \in \{19, 31, 49, 61, 79, 109\}$ then all values of $n < q$ satisfying $n^2+n+1 \equiv 0 \bmod q$ are as listed in Table 4B.

In each such case there must be an orbit of length $[G\colon C_G(\tau)]$, i.e. of type 4 or 5 according as $q \equiv 1$ or 3 mod 4. (For if τ is a perspectivity, as must be the case when n is a non-square, then the centre of τ is fixed by $C_G(\tau)$. If τ is a Baer collineation, $n = 25$, $q = 31$, then the dihedral group $C_G(\tau)$ of order 32 must fix at least one of the 31 points of the Baer subplane Fix(τ)).

Since $[G\colon C_G(\tau)] \le n^2+n+1$, Table 4B shows that $(n,q) = (25,31)$, (32,49) or (47,61) and the number of points remaining is smaller than any of the permissible orbit lengths. This together with (38) yields

$$(40) \qquad n_{10} = n_{11} = n_{12} = 0.$$

Any orbit is therefore of type 4 or 5, and by considering the length of an orbit we see that $w = 1$. If G_1 is of type 4 then τ fixes $\frac{1}{2}(q+1)$ points of Π, while if G_1 is of type 5 then τ fixes $\frac{1}{2}(q-1)$ or $\frac{1}{2}(q+3)$ points of Π according as $q \equiv 1$ or 3 mod 4. Therefore $\frac{1}{2}(q+3) \ge n+1$, i.e. $q \ge 2n-1$, and

$$n^2+n+1 = [G\colon G_1] \ge \tfrac{1}{2}q(q-1) \ge 2n^2-5n+2,$$

contrary to (26). □

5. Further Results

5.1 COROLLARY (Foulser-Johnson [4]). *Any projective translation plane of order q^2 admitting $G \cong$ PSL$(2,q)$ for q odd is Desarguesian.*

Proof. Suppose that Π is a non-Desarguesian projective plane of order q^2 admitting $G \cong$ PSL$(2,q)$ where q is odd. Then G fixes a line of Π, so Theorem 1.2 gives $q \in \{5, 9\}$. By [14,Theorem 3.3] we have $q = 9$. By [14,Remark 3.1] and the remarks thereafter, G is contained in the translation complement of Π, i.e. G fixes an antiflag of Π, contrary to 1.2. □

5.2 COROLLARY. *Suppose that a projective plane* Π *of order* q^2 *admits a collineation group* $G \cong \mathrm{PSL}(2,q)$ *(where* q *is odd), and that* G *leaves invariant a proper subplane* Π_0. *If* $q \neq 5, 9$ *then the following assertions must hold.*

(i) Π_0 *is a Desarguesian Baer subplane of* Π, *on which* G *acts faithfully.*

(ii) G *acts irreducibly on* Π *for* $q > 3$, *and fixes a triangle but no point or line of* Π *for* $q = 3$.

(iii) The involutions in G *act as homologies of* Π.

(iv) Π_0 *is the unique* G*-invariant proper subplane of* Π.

(v) Suppose further that $G \trianglelefteq N$, *where* N *is a collineation group of* Π. *Then* $N/\mathrm{C}_N(G)$ *is isomorphic to a subgroup of* $\mathrm{P\Gamma L}(2,q)$; N *leaves* Π_0 *invariant;* $\mathrm{C}_N(G)$ *is the kernel of the action of* N *on* Π_0, *and* $|\mathrm{C}_N(G)| \mid q(q-1)$.

For $q = 5$, *conclusions (i)–(v) hold under the additional hypothesis that either (i) or (ii) holds. For* $q = 9$, *conclusions (ii)–(v) hold under the additional hypothesis that (i) holds.*

Proof. (i). If $q = 3$ then Π is Desarguesian or a Hughes plane of order 9 by 2.7, and (i) follows easily. Hence we may assume that $q > 3$ and G is simple. By Theorem 1.2, G acts faithfully on Π_0. If Π_0 is a Baer subplane of Π then we are done by Theorem 2.10. Thus we may assume that Π_0 is of order $n < q$, and the possibilities for (n,q) are limited by Theorem 1.1. If $q = 5$ then G acts reducibly on Π_0 of order 4, contrary to the hypothesis; if $q = 9$ then (i) holds by the additional hypothesis. Therefore $q = 7$ and $n = 2$ or 4. Now G contains an element g of order 4, and g^2 induces an elation of Π_0. However g^2 is a homology of Π by 2.5(ii), a contradiction.

(ii). Follows from (i) and 2.10, since every point of Π outside Π_0 lies on a unique line of Π_0, and dually.

(iii). Follows from (i) and Proposition 2.6.

(iv). By (iii), the involutions in G induce homologies of any G-invariant subplane of Π. However, the centres of these homologies generate Π_0.

(v). N acts on G by conjugation, inducing an automorphism group $N/\mathrm{C}_N(G) \subseteq \mathrm{Aut}(G) \cong \mathrm{P\Gamma L}(2,q)$. Let $g \in \mathrm{C}_N(G)$, and let X be a point of Π_0 such that $|X^G| = q+1$. Since g commutes with G_X and X is the unique point of Π_0 fixed by G_X, we have $X^g = X$. Likewise g fixes each point of X^G, and since X^G generates Π_0 we have $\mathrm{C}_N(G) \subseteq N_0$ where N_0 is the kernel of the action of N on Π_0. Now N_0, G are both normal in N, so the commutator subgroup $[N_0, G] \subseteq N_0 \cap G = 1$, i.e. $N_0 \subseteq \mathrm{C}_N(G)$. The remaining assertion follows by 2.2. □

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