

PLANES, SEMIBIPLANES AND RELATED COMPLEXES

G. ERIC MOORHOUSE*

Abstract. We observe a connection between the problem of lifting semiplanes to projective planes, and the 1-cohomology of a related cell complex. Examples are provided using the translation planes of order 16.

1. Lifting ‘quotients’ of planes. Given a projective plane Π with an automorphism group $G \leq \text{Aut } \Pi$, the G -orbits on the points and lines of Π form a tactical decomposition of Π , and the quotient structure, denoted Π/G , has as its points and blocks, the G -orbits on the points and lines of Π (see [1]).

In general $\text{Aut } \Pi$ may have more than one conjugacy class of subgroups isomorphic to G , and we must specify the embedding $\theta : G \rightarrow \text{Aut } \Pi$, and denote the corresponding quotient by Π/θ when the expression Π/G is ambiguous. We call two such pairs (Π, θ) , (Π', θ') **equivalent** if there is an isomorphism $\psi : \Pi \rightarrow \Pi'$ intertwining θ and θ' , i.e. $\theta'(g) \circ \psi = \psi \circ \theta(g)$ for all $g \in G$. Clearly $\Pi/\theta \cong \Pi'/\theta'$ in such a case.

We wish to examine the reverse problem: that of lifting a given object Σ to a projective plane of given order n with prescribed automorphism group G , such that $\Pi/G \cong \Sigma$. We of course assume that Σ satisfies certain necessary (in general not sufficient) conditions for such a lifting to occur. (These conditions amount to certain dot product relations for the rows, and columns, of the incidence matrix defining Σ ; see [1, p.17].) Furthermore, given Σ and G , we wish to know how many isomorphism classes of planes are so obtainable as preimages of Σ ; or better yet, how many equivalence classes of pairs $(\Pi, \theta : G \rightarrow \text{Aut } \Pi)$ yield $\Pi/\theta \cong \Sigma$. In general this is a very difficult problem. In this paper we consider only the case $|G|$ is prime.

2. The case $|G| = 2$. An **elation semiplane** of order n (where n is necessarily even) is an incidence structure of points and blocks, whose incidence matrix may be partitioned in the form

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

where each A_{ij} is a permutation matrix of order $n/2$, and for $i \neq j$ each row of $(A_{i1} \ A_{i2} \ \cdots \ A_{in})$ has dot product 2 with each row of $(A_{j1} \ A_{j2} \ \cdots \ A_{jn})$. For instance the unique elation semiplane of order 4 has incidence matrix given by Figure 1.

* Department of Mathematics, University of Wyoming, Laramie WY 82071–3036. Supported by NSERC of Canada under a postdoctoral fellowship.

FIGURE 1. Incidence matrix
of the unique elation
semiplane Σ of order 4,
with points labelled 1, 2, ..., 8;
blocks labelled a, b, ..., h

	a	b	c	d	e	f	g	h
1	1		1		1		1	
2		1		1		1		1
3	1		1			1		1
4		1		1			1	
5	1			1		1		1
6		1			1		1	
7	1		1		1		1	
8		1		1		1		1

The motivation for this definition is that if Π is a projective plane of order n with an involutory elation τ , then the τ -orbits of length 2 on the points and lines of Π , inherit an incidence structure denoted Π/τ , which is an elation semiplane of order n as defined above (see [3]). For instance, the unique projective plane of order 4 has an incidence matrix given by Figure 2, where the operation which fixes the first five rows (respectively, columns) and permuting the remaining rows (resp., columns) in consecutive pairs, is an involutory elation, such that the resulting elation semiplane is clearly the example of Fig. 1.

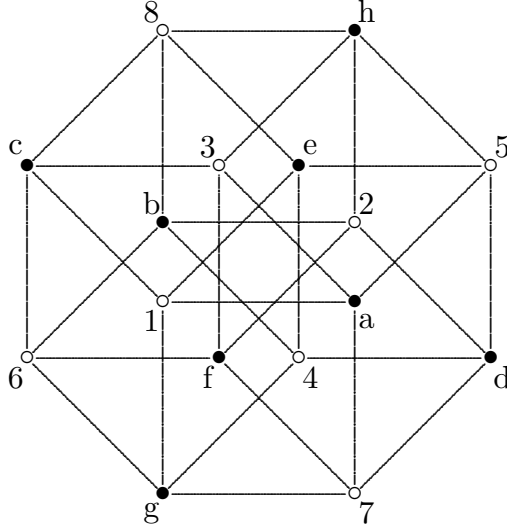
FIGURE 2.
Incidence matrix
of the projective
plane of order 4

1	1	1	1	1																
1					1	1	1	1												
1									1	1	1	1								
1													1	1	1	1				
1																	1	1	1	1
1	1																			
1		1																		
1			1																	
1				1																
1					1															
1						1														
1							1													
1								1												
1									1											
1										1										
1											1									
1												1								
1													1							
1														1						
1															1					
1																1				
1																	1			
1																		1		
1																			1	
1																				1

Similar definitions of homology and Baer semiplanes are possible, where τ is replaced by an involutory homology or Baer collineation. For the remainder of this Section, Σ denotes an elation (or homology, or Baer) semiplane. We wish to determine all (if any) equivalence classes of pairs (Π, τ) such that $\Pi/\tau \cong \Sigma$.

To accomplish this we define a cell complex $\Gamma = \Gamma(\Sigma)$ of rank 2 (all of whose 2-cells are squares) as follows. The vertices (0-cells) of Γ are the points and blocks of Σ . The edges (1-cells) of Γ are the flags of Σ . The faces (2-cells) of Γ are the **digons** of Σ , i.e. the 4-tuples (P, Q, L, M) in Σ where the points $P \neq Q$ both lie on the blocks $L \neq M$. Incidence in Γ is naturally induced from Σ . Note that Γ is nothing but the incidence graph of Σ , with the digons ‘shaded in’. For example the semiplane of Fig. 1 yields the 2-skeleton of the tesseract (4-cube), shown in Figure 3 (although for clarity, we have not shaded the faces of Γ , namely all 4-gons of the figure).

FIGURE 3.
The cell complex $\Gamma(\Sigma)$
for Σ as in Fig. 1



Let $F = \text{GF}(2)$, and let $C^i = C^i(\Gamma, F)$ be the F -space freely spanned by the i -cells of Γ , and let $\delta : C^i \rightarrow C^{i+1}$ be the coboundary operator. Note that each flag of Σ (i.e. edge of Γ) must ‘lift’ to either $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the incidence matrix of Π . The lifting from Σ to Π is completely determined by specifying those flags of Σ which ‘lift’ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and this list of flags of Σ corresponds to an element $\alpha \in C^1$. For the example of Figures 1 and 2, we have

$$\alpha = 3c + 3h + 4d + 4g + 5d + 5e + 6c + 6f + 7f + 7g + 8e + 8h \in C^1.$$

Clearly not every $\alpha \in C^1$ lifts Σ to a plane. We say that $\alpha \in C^1$ is **admissible** if $\delta\alpha = \sigma$, where $\sigma \in C^2$ is defined by $\sigma = \sum\{S : S \text{ is a face of } \Gamma\} \in C^2$. It is not hard to see that

PROPOSITION 1. ‘Liftings’ from Σ to planes, correspond bijectively to admissible elements of C^1 .

(Here a subtle point is that a plane Π with involutory collineation τ , is uniquely determined by the incidences between those of its points and lines which are not fixed by τ . This is why in this case we may safely deviate from the definition of Σ in Section 1 by disregarding the point and line orbits of size 1.)

Next observe that if $\alpha, \beta \in C^1$ are admissible, then $\delta(\alpha + \beta) = \sigma + \sigma = 0$. Thus

PROPOSITION 2. The set of admissible elements of C^1 is either empty, or a coset of $Z^1 = \ker \delta|_{C^1} : C^1 \rightarrow C^2$.

For admissible $\alpha \in C^1$, let Σ^α denote the corresponding plane obtained by lifting Σ . Clearly it is possible that $\Sigma^\beta \cong \Sigma^\alpha$ for admissible $\beta \neq \alpha$. Indeed in the above example, switching rows 6 and 7 of Fig. 2 corresponds to replacing α by $\alpha + 1a + 1c + 1e + 1g \in C^1$. More generally, we have:

PROPOSITION 3. Let $\alpha, \beta \in C^1$ be admissible. If $\alpha \equiv \beta \pmod{B^1 = \delta C^0 \leq C^1}$ then $\Sigma^\alpha \cong \Sigma^\beta$.

PROPOSITION 4. Given Σ , the equivalence classes of pairs (Π, τ) such that $\Pi/\tau \cong \Sigma$ are in bijective correspondence with the orbits of $\text{Aut } \Sigma$ on $\{\alpha \in C^1 : \alpha \text{ admissible}\}/B^1$.

Note regarding this notation: In view of Prop. 2, $\{\alpha \in C^1 : \alpha \text{ admissible}\}/B^1$ is either empty, or a coset of Z^1/B^1 in C^1/B^1 . In the latter case, we warn the reader that although $\{\alpha \in C^1 : \alpha \text{ admissible}\}/B^1$ and $H^1 = Z^1/B^1$ have the same size and are both stable under $\text{Aut } \Sigma$, the representation of $\text{Aut } \Sigma$ on H^1 may not be equivalent to the action on $\{\alpha \in C^1 : \alpha \text{ admissible}\}/B^1$.

COROLLARY. If $H^1(\Gamma, F) = 0$ then there is at most one equivalence class of pairs (Π, τ) such that $\Pi/\tau \cong \Sigma$.

The proof of the main result of [4] shows the following.

THEOREM. If $\Sigma \cong \Pi/\tau$ where Π is Desarguesian of prime order, and τ is an involutory collineation, and $\Gamma = \Gamma(\Sigma)$, then $H^1(\Gamma, F) = 0$. Thus Σ lifts uniquely to within equivalence.

We expect that our new interpretation of the lifting problem involving the cell complex Γ will lead to a generalization of this Theorem to all prime powers, and to other similar lifting results.

3. Translation planes of order 16. Using computer, we have produced all elation semiplanes arising from the eight translation planes of order 16 (classified by Dempwolff and Reifart [2]) and their duals; checked this list of semiplanes for duplications (isomorphisms and dualities); and lifted each semiplane in our list, verifying that each projective plane so obtained is a translation plane or dual translation plane, and so (according to the classification [2]) is not a new plane. (Note: we did *not* classify all elation semiplanes of order 16.) The results are as follows.

Exactly 56 isomorphism classes of elation semiplanes were thus obtained, namely $\Sigma_1, \Sigma_2, \dots, \Sigma_{31}$, and $\Sigma_7^*, \Sigma_8^*, \dots, \Sigma_{31}^*$, where $*$ indicates dual ($\Sigma_1, \Sigma_2, \dots, \Sigma_6$ are self-dual). We found $\dim H^1(\Gamma(\Sigma_i), F) = 4$ for $i = 1$; 2 for $i = 2$; 1 for $i \in \{3, 4, 7, 8\}$; and 0 for $i \in \{5, 6, 9, 10, 11, \dots, 31\}$.

- (i) The automorphism group of the *Desarguesian* plane of order 16 has a single conjugacy class of involutory elations, yielding the elation semiplane Σ_5 , which lifts uniquely (to within isomorphism).
- (ii) The automorphism group of the *Hall* plane of order 16 has three conjugacy classes of involutory elations, consisting of 75 translations, 180 translations, and 5 shears, respectively. These yield the semiplanes Σ_9, Σ_{10} and Σ_{11} resp., each of which lifts uniquely (i.e. to within isomorphism).

- (iii) The *semifield plane* of order 16 with kernel $\text{GF}(4)$ has six classes of involutory elations, consisting of 6 translations, 9 translations, 96 translations, 144 translations, 6 shears, and 9 shears. These yield the semibiplanes $\Sigma_2, \Sigma_3, \Sigma_{12}, \Sigma_{13}, \Sigma_{12}^*, \Sigma_{13}^*$ resp. Each of these lifts uniquely, with the exception of Σ_2 , which also lifts to the semifield plane with kernel $\text{GF}(2)$.
- (iv) The *semifield plane* of order 16 with kernel $\text{GF}(2)$ has nine classes of involutory elations, consisting of 3 translations, 3 translations, 9 translations, 48 translations, 48 translations, 144 translations, 3 shears, 3 shears, and 9 shears. These yield $\Sigma_2, \Sigma_6, \Sigma_4, \Sigma_{14}, \Sigma_{15}, \Sigma_{16}, \Sigma_{14}^*, \Sigma_{15}^*$ and Σ_{16}^* resp. All of these lift uniquely, except Σ_2 (see (iii) above).
- (v) The *derived semifield plane* has six classes of involutory elations, of size 6, 9, 24, 36, 72 and 108 (all consisting of translations). These yield $\Sigma_{17}, \Sigma_7, \Sigma_{18}, \Sigma_{19}, \Sigma_{20}$ and Σ_{21} resp. Each of these lifts uniquely, except Σ_7 , which also lifts to the Lorimer-Rahilly plane.
- (vi) The *Lorimer-Rahilly plane* has five classes of involutory elations: 3 translations, 42 translations, 42 translations, 168 translations and 3 shears. These yield $\Sigma_1, \Sigma_7, \Sigma_{22}, \Sigma_{23}$ and Σ_{24} resp. The latter three semibiplanes lift uniquely, while Σ_1 lifts to both Σ_1 and its dual; and Σ_7 lifts to both the Lorimer-Rahilly plane and the derived semifield plane.
- (vii) The *Johnson-Walker plane* has five classes of involutory elations: 21 translations, 24 translations, 42 translations, 168 translations and 3 shears. These yield $\Sigma_{25}, \Sigma_{26}, \Sigma_8, \Sigma_{27}$ and Σ_{28} resp. Each of these lifts uniquely, except Σ_8 , which also lifts to the Dempwolff plane.
- (viii) The *Dempwolff plane* has four classes of involutory elations, of size 15, 15, 45 and 180 (all consisting of translations). These yield $\Sigma_8, \Sigma_{29}, \Sigma_{30}$ and Σ_{31} resp. Each of these lifts uniquely, except Σ_8 (see (vii) above).

4. The case $|G| = p$ is an odd prime. Suppose that G is generated by a quasiperspective (i.e. perspectivity or Baer collineation) of odd prime order p . Again we may ask which semisymmetric designs Σ occur as quotients of planes by such collineation groups. Define $\Gamma = \Gamma(\Sigma)$ just as before, using digons as 2-cells. Let $F = \text{GF}(p)$ and define $C^i = C^i(\Gamma, F)$ and δ as before. The obvious analogues of Propositions 3 and 4 are valid. The analogue of Prop. 1 holds only if we redefine $\alpha \in C^1$ as **admissible** if $\delta\alpha$ has maximal weight in C^2 , i.e. if each face of Γ occurs in $\delta\alpha$ with nonzero coefficient. It is not clear how to utilize this condition; Prop. 2 and the Corollary have no direct analogue for this situation. We have not found an interpretation for $H^1(\Gamma, F)$ which is relevant to the lifting problem in this case.

REFERENCES

- [1] P. DEMBOWSKI, *Finite Geometries*, Springer-Verlag, Berlin, 1968.
- [2] U. DEMPWOLFF AND R. REIFART, *The classification of translation planes of order 16, I*, *Geom. Ded.*, **15** No. 2 (1983), 137–153.
- [3] D. R. HUGHES, *Biplanes and semiplanes*, in *Combinatorics*, Lecture Notes in Mathematics No. 686, pp. 55–58, Springer-Verlag, Berlin, 1978.
- [4] G. E. MOORHOUSE, *On the construction of finite projective planes from homology semiplanes*, submitted.