

Groups of projective planes with differing numbers of point and line orbits

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Abstract

We give examples of infinite projective planes with collineation groups having differing numbers of orbits on points and on lines, solving a problem posed by Cameron [9] and attributed to Kantor. Some of our examples are Desarguesian.

Keywords: projective plane, collineation group, orbit, skew field

1. Introduction and History

The origin of this problem can be traced to a paper of Brauer [5], where (in Section 6, Lemma 3) he proves a result, that combined with, for example, the results of Bose [4], demonstrates that every group of collineations of a finite projective plane has equally many orbits on points and on lines. (The use of the incidence matrix goes back at least to Levi [24], but it is unknown to these authors when the incidence matrix of a finite projective plane was first observed to be invertible. The modern proof of Fisher's 1940 inequality for 2-designs [19] is due to Bose [4], and the modern proof of Baer's 1946 results [1] on polarities of projective planes is due to Devidé [15]. Certainly, that the incidence matrix of a finite projective plane is invertible was known already to Bruck and Ryser [6] and Bose [4].) The result that every group of collineations of a finite projective plane has equally many orbits on points and on lines was rediscovered in the more general context of finite symmetric designs by Parker [26], Hughes [21], Dembowski [14], as the theorem that every automorphism group of a finite symmetric design has equally many orbits on points and on blocks, and is therefore sometimes referred to as the Dembowski-Hughes-Parker theorem. Block [3] showed that every group of automorphisms of a finite 2-design has at least as many orbits on blocks as on points, from which the Dembowski-Hughes-Parker theorem for finite symmetric designs follows, so this result is often (at least implicitly) referred to as Block's Lemma.

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The question of whether or not every group of collineations of an infinite projective plane must have equally many orbits on points and on lines seems to have first been raised in print by Cameron [7], with the same author returning to the topic seven years later at a conference in Capri in 1991, published in Cameron [9], where the question is attributed to W.M. Kantor. The related question of whether or not every automorphism group of an infinite Steiner system has at least as many orbits on blocks as on points, also raised in Cameron [7], has been settled in the negative by Evans [18], with further examples (for designs rather than Steiner systems) in Camina [11] and Webb [32], who were building on earlier work of Evans [17]. The question was settled in the affirmative for infinite Steiner *triple* systems by Cameron at a conference in Guildford in 1991, published in Cameron [8], with further results in a positive direction appearing in Webb [31], with the confusion arising leading to an attempted redefinition of an infinite design in Cameron and Webb [10]. But, until now, the question for infinite projective planes has remained open. This paper settles that question in the negative.

The origin of the first examples we shall present can be traced back to a question of Emil Artin in the December 1946 Princeton University Bicentennial Conference on Problems in Mathematics. To quote from the review MR1791874 by John W. Dawson, Jr. in Mathematical Reviews, “Due to lack of funds, no proceedings of the Princeton Bicentennial Conference on Problems of Mathematics ever appeared. The only publication to chronicle the event was a small pamphlet, published the following year, entitled Problems of mathematics, which listed the eighty-two invited participants, described the ten sessions into which the congress was subdivided, and gave brief summaries of the discussions that followed the invited addresses in each session.” Fortunately, this pamphlet (reviewed as MR0020520) was reprinted by the American Mathematical Society [16]. That question (which is referred to on page 312 of the reprint) is, given an extension of skew fields, whether or not the left and right dimension must be equal. This problem was solved by Cohn [12], where examples were given with one dimension finite and the other infinite, and then Schofield [28] showed that for any finite pair (m, n) of integers, each greater than one, there is an extension of skew fields with left rank m and right rank n . More explicit constructions appear in Treur [30]. See also the books by Schofield [29] and Cohn [13], particularly Theorems 5.9.1 and 5.9.2 in the latter.

The idea to use these extensions of skew fields constructed by Schofield to solve Kantor’s question was inspired by Salzmann [27], Proposition 1.6. (That result in turn has its origin in Barlotti [2].) Maier and Stroppel [25] come very close to these examples, as do Jha and Johnson [22] (see their Theorem 4). Our construction in Section 2 yields collineation groups of Desarguesian planes with two point orbits and at least three line orbits; other numbers of point and line orbits for groups of Desarguesian projective planes may be obtained by the same method.

We also construct collineation groups of projective planes with *arbitrary* numbers of point and line orbits in Section 3, using ideas inspired by Hughes [20].

2. The Desarguesian examples

We need the group $\text{PGL}(3, K) = \text{GL}(3, K)/Z(\text{GL}(3, K))$, for a division ring K where, for clarity, we point out that $Z(\text{GL}(3, K)) = \{zI : z \in Z(K)^\times\}$, where $Z(K)$ is the center of K , and $Z(K)^\times = Z(K^\times)$ is the multiplicative group of nonzero elements. We will use homogeneous coordinates for the points of the Desarguesian projective plane $\text{PG}(2, K)$ over K , arising from a three-dimensional left K vector space, and denoted by (a, b, c) , where $a, b, c \in K$, not all zero. Homogeneity of coordinates means that the triple $(\lambda a, \lambda b, \lambda c)$ represents the same point as (a, b, c) whenever $\lambda \in K^\times$. Lines will be represented as column vectors $(l, m, n)^T$ where $l, m, n \in K$, not all zero; here $(l\lambda, m\lambda, n\lambda)^T$ represents the same line as $(l, m, n)^T$ whenever $\lambda \in K^\times$. The point (a, b, c) lies on the line $(l, m, n)^T$ iff $al + bm + cn = 0$. We note that $\text{PGL}(3, K)$ acts transitively on both the points and the lines of $\text{PG}(2, K)$; the action of $A \in \text{PGL}(3, K)$ on points is given by $(a, b, c) \mapsto (a, b, c)A$, and the action on lines is given by $(l, m, n)^T \mapsto A^{-1}(l, m, n)^T = [(l, m, n)A^{-T}]^T$.

Now suppose the skew field K is a proper subring of the skew field L , so that $\pi' = \text{PG}(2, K)$ is a subplane of $\pi = \text{PG}(2, L)$, and $\text{PGL}(3, K)$ is a subgroup of $\text{PGL}(3, L)$, stabilizing π' . We examine the orbits of $G = \text{PGL}(3, K)$, first on the points of π . (By transposing, our arguments yield similarly the orbits on lines of π .) Consider the G -orbit of a point $P = (a, b, c)$ of π . We may assume $a \in K^\times$; otherwise apply a permutation matrix in G to move a nonzero value to the first coordinate. Using homogeneity of coordinates, we may further assume $a = 1$.

If K has right dimension 2 over L , say with basis $\{1, \alpha\}$, then $P = (1, b_1 + \alpha b_2, c_1 + \alpha c_2)$ for some $b_i, c_i \in K$. If $b_2 = c_2 = 0$ then P lies in π' and an element of G exists mapping $P \mapsto (1, 0, 0)$. Otherwise, without loss of generality $b_2 \neq 0$ and the matrix

$$\begin{bmatrix} 1 & -b_1 b_2^{-1} & b_1 b_2^{-1} c_2 - c_1 \\ 0 & b_2^{-1} & -b_2^{-1} c_2 \\ 0 & 0 & 1 \end{bmatrix}$$

maps $P \mapsto (1, \alpha, 0)$. The latter point lies outside π' and since G preserves π' , G has two orbits on points of π , those points (a, b, c) in π' (characterized by the property that a, b, c span a one-dimensional right K -subspace of L) and those points (a, b, c) of π outside π' (for which a, b, c span L as a right vector space over K).

Theorem 2.1. *Let L be a skew field containing the skew field K with the left dimension of L over K being greater than 2 and the right dimension of L over K being 2. Let π be the (Desarguesian) projective plane afforded by a three-dimensional left L vector space, and consider the natural action of $G = \text{PGL}(3, K)$ on π (by right multiplication). Then G has 2 orbits on points of π , and at least 3 orbits on lines of π .*

PROOF. G acts transitively on the points and lines of a subplane π' of π (coordinatized by K). The stabilizer of a line ℓ of π has two orbits on the points of ℓ ,

since, G_ℓ^ℓ is permutationally isomorphic to $\text{PGL}(2, K)$ acting on $\text{PG}(1, L)$, and, for any two elements b, c of $L - K$, right linearly independent over K , $\{1, b, c\}$ is right linearly dependent over K . Thus there is an element of $\text{PGL}(2, K)$ taking $(1, b)$ to $(1, c)$, namely $\begin{bmatrix} s & -t \\ 0 & -r \end{bmatrix}$, where $r, s, t \in K$ with $r + sb + tc = 0$. Similarly, if $P(a, b, c)$ is a point of π , then a, b and c are right linearly dependent over K , so there exist $r, s, t \in K$ (not all zero) such that $ra + sb + tc = 0$; that is, P lies on the line $[r, s, t]$ of π' . Hence G has two orbits on points on π . Now take left linearly independent elements r, s, t of L over K . Then $[r, s, t]$ is a line of π that is disjoint from π' . So G has at least 3 orbits on lines of π , as each of the (extended) lines of π' , the lines of π meeting π' in a single point, and the lines of π disjoint from π' forms a G -invariant set.

Again, consider an extension $K \subseteq L$ having left degree m and right degree n , with $G = \text{PGL}(3, K)$ acting on $\pi = \text{PG}(2, L)$ as above. Closer inspection reveals that for $(m, n) = (3, 2)$ we have exactly 2 point orbits and 3 line orbits. In this case, the orbit of a point (a, b, c) depends just on the dimension (1 or 2) of the right subspace of L spanned by a, b, c over K ; and the orbit of a line $[r, s, t]$ depends just on the dimension (1, 2 or 3) of the left subspace of L spanned by r, s, t over K . However, for $(m, n) = (4, 2)$, G has two orbits on points and infinitely many orbits on lines. Does there exist a collineation group of a Desarguesian plane with exactly two point orbits and four line orbits? Still more startlingly, does there exist a Desarguesian plane with a group acting transitively on points but intransitively on lines? These questions we have not addressed.

The corresponding questions for arbitrary (not necessarily Desarguesian) projective planes, will be answered in Section 3.

3. Arbitrary numbers of point and line orbits

Fix a group G (multiplicative with identity element 1) with the following three properties:

- (G1) G is a nonabelian infinite group.
- (G2) Every conjugacy class in G other than $\{1\}$ has cardinality $|G|$.
- (G3) Every element of G has at most one square root in G .

For every infinite cardinal number C , there exists such a group of cardinality C ; take, for example, a free group on C generators. (When C is countable, 2 generators suffice.) Observe the following easy consequence of (G3):

- (G3') For all $g, h \in G$, there is at most one $x \in G$ satisfying $xhx = g$.

A projective plane of (infinite) order $|G|$ has point and line sets both of cardinality $|G|$, so the number of orbits on points or on lines is at most $|G|$. We show that this necessary condition is also sufficient:

Theorem 3.1. *Let A and B be nonempty sets with $|A|, |B| \leq |G|$ where G satisfies (G1)–(G3). Then there exists a projective plane π of order $|G|$ with a group of collineations isomorphic to G , having exactly $|A|$ point orbits and $|B|$ line orbits.*

Fix a pair of nonempty sets A, B with $|A|, |B| \leq |G|$. We also require an indexed family of subsets $D_{a,b} \subset G$ for $(a, b) \in A \times B$ satisfying the following:

- (D1) For all $b_1, b_2 \in B$ and $g \in G$, there exists $a \in A$ and a pair of elements $d_1 \in D_{a,b_1}, d_2 \in D_{a,b_2}$ such that $g = d_1^{-1}d_2$. The triple (a, d_1, d_2) is unique whenever $(b_1, g) \neq (b_2, 1)$.
- (D2) For all $a_1, a_2 \in A$ and $g \in G$, there exists $b \in B$ and a pair of elements $d_1 \in D_{a_1,b}, d_2 \in D_{a_2,b}$ such that $g = d_2d_1^{-1}$. The triple (b, d_1, d_2) is unique whenever $(a_1, g) \neq (a_2, 1)$.

These conditions are slightly redundant, in that the uniqueness in (D2) follows from the uniqueness in (D1). Indeed, the following consequence of (D1)–(D2) expresses the ‘uniqueness’ part of both (D1) and (D2):

- (D3) Whenever $d_1^{-1}d_2 = d_3^{-1}d_4$ where $d_1 \in D_{a,b}, d_2 \in D_{a,b'}, d_3 \in D_{a',b}$ and $d_4 \in D_{a',b'}$, we have $(a, d_1, d_2) = (a', d_3, d_4)$ or $(b, d_1, d_3) = (b', d_2, d_4)$.

Assuming such subsets $D_{a,b} \subset G$ exist (satisfying (D1)–(D2), and therefore also (D3)), then a projective plane π as described by Theorem 3.1 is constructed having the elements of $G \times A$ as points, and elements of $G \times B$ as lines. (If A and B are not disjoint, then context will determine whether a pair (g, a) denotes a point or a line.) Incidence in π is defined as follows: a point $(x, a) \in G \times A$ lies on a line $(y, b) \in G \times B$ iff $xy^{-1} \in D_{a,b}$. The group G acts on π as follows: the element $g \in G$ permutes points via $(x, a) \mapsto (xg, a)$ and lines via $(y, b) \mapsto (yg, b)$. It is a simple exercise to check that π is a projective plane, and that G preserves incidence in π , so G is a group of collineations of π with $|A|$ regular point orbits $G \times \{a\}$, and $|B|$ regular line orbits $G \times \{b\}$. In order to prove Theorem 3.1, it suffices to show that such an indexed family of subsets of G exists.

Let \mathcal{S} be the collection of all indexed families of subsets $D_{a,b} \subset G$ satisfying (D3). We will obtain the required solution of (D1)–(D2) as the union of a certain chain in \mathcal{S} . The ability to suitably extend solutions of (D3) to obtain the desired chains, follows from

Proposition 3.2. *Consider a family $\{D_{a,b} : (a, b) \in A \times B\}$ of subsets of G of cardinality $|D_{a,b}| < |G|$ satisfying (D3).*

- (i) *Let $g \in G, a_1 \in A$ and $b_1, b_2 \in B$ where $b_1 \neq b_2$. Suppose there is no triple (a, d_1, d_2) with $a \in A, d_i \in D_{a,b_i}$ satisfying $g = d_1^{-1}d_2$. Then there exists $x \in G$ such that the family of subsets*

$$\tilde{D}_{a,b} = \begin{cases} D_{a_1,b_1} \cup \{x\}, & \text{if } (a, b) = (a_1, b_1); \\ D_{a_1,b_2} \cup \{xg\}, & \text{if } (a, b) = (a_1, b_2); \\ D_{a,b}, & \text{otherwise} \end{cases}$$

satisfies (D3), and yields the missing expression $g = x^{-1}(xg)$.

- (ii) Let $g \in G$, $a_1 \in A$ and $b_1 \in B$. Suppose there is no triple (a, d_1, d_2) with $a \in A$, $d_i \in D_{a,b_1}$ satisfying $g = d_1^{-1}d_2$. Then there exists $x \in G$ such that the family of subsets

$$\tilde{D}_{a,b} = \begin{cases} D_{a_1,b_1} \cup \{x, xg\}, & \text{if } (a, b) = (a_1, b_1); \\ D_{a,b}, & \text{otherwise} \end{cases}$$

satisfies (D3), and yields the missing expression $g = x^{-1}(xg)$.

- (iii) Let $g \in G$, $a_1, a_2 \in A$ where $a_1 \neq a_2$, and $b_1 \in B$. Suppose there is no triple (b, d_1, d_2) with $b \in B$, $d_i \in D_{a_i,b}$ satisfying $g = d_2d_1^{-1}$. Then there exists $x \in G$ such that the family of subsets

$$\tilde{D}_{a,b} = \begin{cases} D_{a_1,b_1} \cup \{x\}, & \text{if } (a, b) = (a_1, b_1); \\ D_{a_2,b_1} \cup \{gx\}, & \text{if } (a, b) = (a_2, b_1); \\ D_{a,b}, & \text{otherwise} \end{cases}$$

satisfies (D3), and yields the missing expression $g = (gx)x^{-1}$.

- (iv) Let $g \in G$, $a_1 \in A$ and $b_1 \in B$. Suppose there is no triple (b, d_1, d_2) with $b \in B$, $d_i \in D_{a_i,b}$ satisfying $g = d_2d_1^{-1}$. Then there exists $x \in G$ such that the family of subsets

$$\tilde{D}_{a,b} = \begin{cases} D_{a_1,b_1} \cup \{x, gx\}, & \text{if } (a, b) = (a_1, b_1); \\ D_{a,b}, & \text{otherwise} \end{cases}$$

satisfies (D3), and yields the missing expression $g = (gx)x^{-1}$.

PROOF. Assume the hypotheses of (i). In order that the sets $\tilde{D}_{a,b}$ satisfy (D3), it suffices to choose $x \in G$ such that for all $(a, b) \in A \times B$ and $d_1 \in D_{a,b}$, $d_2 \in D_{a,b_1}$, $d'_2 \in D_{a,b_2}$, $d_3 \in D_{a_1,b}$, $d_4, d_5 \in D_{a_1,b_1}$, $d'_4, d'_5 \in D_{a_1,b_2}$,

$$\begin{cases} x \neq d_1, d_1g^{-1}, d_3d_1^{-1}d_2, d_3d_1^{-1}d'_2g^{-1}; \\ xd_4^{-1}x \neq d_5, d'_4g^{-1}; \\ xg(d'_4)^{-1}x \neq d'_5. \end{cases}$$

By (G1) and (G3'), and the condition $|D_{a,b}| < |G|$, each of these restrictions excludes fewer than $|G|$ of the elements of G , so such an element $x \in G$ can be found. (In the general case, we require the Axiom of Choice. As usual, in the countable case an explicit enumeration of G is available so x is found by finite search as the least feasible element of G , thereby avoiding the Axiom of Choice.) We freely use the fact that for any sets X and Y , not both finite, $|X \times Y| = |X \cup Y| = \max\{|X|, |Y|\}$; see e.g. [23, p.40]. The conclusions of (i) follow.

Next, assume the hypotheses of (ii). In order that the sets $\tilde{D}_{a,b}$ satisfy (D3), it suffices to choose $x \in G$ such that for all $(a, b) \in A \times B$ and $d_1 \in D_{a,b}$,

$d_2 \in D_{a,b_1}$, $d_3 \in D_{a_1,b}$, $d_4, d_5 \in D_{a_1,b_1}$,

$$\begin{cases} x \neq d_1, d_1g^{-1}, d_3d_1^{-1}d_2, d_3d_1^{-1}d_2g^{-1}; \\ xgx^{-1} \neq d_5d_4^{-1}; \\ xd_4^{-1}x \neq d_5; \\ xgd_4^{-1}x \neq d_5, d_5g^{-1}. \end{cases}$$

First observe that there are $|G|$ choices of $x \in G$ satisfying $xgx^{-1} \notin \{d_5d_4^{-1} : d_4, d_5 \in D_{a_1,b_1}\}$. (This condition holds trivially if $D_{a_1,b_1} = \emptyset$. Otherwise $g \neq 1$ by the hypotheses of (ii), so by (G2) the conjugacy class of g has size $|G|$, whereas $|\{d_5d_4^{-1} : d_4, d_5 \in D_{a_1,b_1}\}| < |G|$.) As in (i), the remaining conditions exclude fewer than $|G|$ possible choices of x , so such an $x \in G$ exists.

Cases (iii) and (iv) are similar to (i) and (ii).

We now prove Theorem 3.1 by transfinite induction; see e.g. [23, pp.161–163] for the requisite background on transfinite induction and recursion. The steps in this induction will be labeled by the set of triples

$$\text{STEPS} = (A \times A \times G) \cup (B \times B \times G)$$

(here we assume that $A \cap B = \emptyset$). Well-order the set of steps as

$$\text{STEPS} = \{\text{STEP}(\alpha) : \alpha < C\}$$

where $C = |\text{STEPS}| = \max\{|A|, |B|, |G|\} = |G|$; see e.g. [23, p.40]. Here C is a cardinal number, namely the least ordinal of cardinality $|G|$; and the index α ranges over all ordinals less than C . An important consequence of this particular choice of well ordering is that for all $\alpha < C$, the subset $\{\text{STEP}(\beta) : \beta < \alpha\} \subset \text{STEPS}$ has cardinality $|\alpha| < C$. For each $(a, b) \in A \times B$, we define the subsets $D_{a,b}(\alpha) \subset G$ recursively for $\alpha < C$. We will show that for each $\alpha < C$ the family of subsets $\{D_{a,b}(\alpha) : (a, b) \in A \times B\}$ satisfies (D3), and moreover

- (0) $D_{a,b}(\beta) \subseteq D_{a,b}(\alpha)$ whenever $\beta \leq \alpha < C$; also $|D_{a,b}(\alpha)| \leq 2|\alpha| < |G|$.
- (1) For every $\beta < \alpha$ such that $\text{STEP}(\beta) = (b_1, b_2, g) \in B \times B \times G$, there exists $a \in A$ and a pair of elements $d_1 \in D_{a,b_1}(\alpha)$, $d_2 \in D_{a,b_2}(\alpha)$ such that $g = d_1^{-1}d_2$.
- (2) For every $\beta < \alpha$ such that $\text{STEP}(\beta) = (a_1, a_2, g) \in A \times A \times G$, there exists $b \in B$ and a pair of elements $d_1 \in D_{a_1,b}(\alpha)$, $d_2 \in D_{a_2,b}(\alpha)$ such that $g = d_2d_1^{-1}$.

For the smallest index $\alpha = 0$, we take $D_{a,b}(0) = \emptyset$ which vacuously satisfies (D3) and (0)–(2).

Next, suppose the ordinal $\alpha < C$ is a successor ordinal, say $\alpha = \beta + 1$. We obtain the family of extended subsets $D_{a,b}(\alpha) \supseteq D_{a,b}(\beta)$ by adjoining at most two more elements, as follows. Recall that $\text{STEP}(\beta) = (a_1, a_2, g)$ or (b_1, b_2, g) ; we consider only the case $\text{STEP}(\beta) = (b_1, b_2, g) \in B \times B \times G$, since the other case is similar. We consider three subcases:

- (i) Suppose $g = d_1^{-1}d_2$ for some $d_i \in D_{a,b_i}(\beta)$ and $a \in A$. In this case, add nothing: simply define $D_{a,b}(\alpha) = D_{a,b}(\beta)$ for all a, b .
- (ii) If (i) fails and $b_1 \neq b_2$, we first choose $a_1 \in A$ arbitrarily. We then obtain the subsets $D_{a,b}(\alpha) \supseteq D_{a,b}(\beta)$ by adjoining one new element for $(a, b) \in \{(a_1, b_1), (a_1, b_2)\}$ according to Proposition 3.2(i), and adding no elements for each of the remaining pairs (a, b) .
- (iii) If (i) fails and $b_1 = b_2$, we first choose $a_1 \in A$ arbitrarily. We then obtain the subsets $D_{a,b}(\alpha) \supseteq D_{a,b}(\beta)$ by adjoining two new elements for $(a, b) = (a_1, b_1)$, and leaving all other subsets the same, according to Proposition 3.2(ii).

Since at most two new elements are adjoined, (0) still holds.

Finally if $\alpha < C$ is a limit ordinal, we set $D_{a,b}(\alpha) = \bigcup_{\beta < \alpha} D_{a,b}(\beta)$, which also satisfies (D3) and (0)–(2). Here the bound $|D_{a,b}(\alpha)| \leq 2|\alpha| < |G|$ follows from the fact that by our construction, $D_{a,b}(\alpha)$ is formed from $D_{a,b}(0) = \emptyset$ by adjoining at most $2|\alpha|$ elements (at most two new elements at each recursive step).

This verifies that the subsets $D_{a,b}(\alpha)$ have the required properties for all $\alpha < C$. Finally, we set $D_{a,b} = \bigcup_{\alpha < C} D_{a,b}(\alpha)$ for all $(a, b) \in A \times B$. This family of subsets of G satisfies (D1)–(D2) by construction, thus completing the proof of Theorem 3.1.

As in the proof of Proposition 3.2, in the countable case the transfinite induction reduces to ordinary induction, using an explicit enumeration of G .

A final remark: We have not addressed the question of conditions on the subsets $D_{a,b}$ under which the group G of Theorem 3.1 may be expected to be the *full* collineation group of the resulting plane π . There is such freedom in choosing the subsets $D_{a,b} \subset G$ that one might hope to be able to force this property to either hold or fail, as desired; this feature is lacking in the Desarguesian construction of Section 2. We leave this as an open project for future investigation.

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