A sufficient condition for an entire function to be a polynomial of degree one

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It is well known that affine transformations of Euclidean spaces transform straight lines into straight lines. In this article, we show that for holomorphic transformations of $\mathbb{C}$, there is a strong converse: it is sufficient to consider the transformation of one straight line. More precisely, if $f : \mathbb{C} \to \mathbb{C}$ is an entire function such that $f(z)$ can take a value on a given straight line L_2 only when z belongs to a certain other straight line L_1, then f must be an affine transformation.[1]

The classical book of R. P. Boas [1], and the monograph of B. Ja. Levin [3] provide extensive results on entire functions, but neither book treats the result of this paper. Some other interesting results in this area have been published in [2], [4], [5].

Lemma. *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that*

$$f(z) \in \mathbb{R} \quad \Rightarrow \quad z \in \mathbb{R} \qquad \forall z \in \mathbb{C}. \tag{1}$$

Then f is a polynomial of the first degree with real coefficients:

$$f(z) = a_1 z + a_0 \qquad \forall z \in \mathbb{C},$$

[1] 1991 Mathematics Subject Classification: 30D20, 30D15.

for some $a_1, a_0 \in \mathbb{R}$.

Proof. Without loss in generality, assume f is a nonconstant entire function. Let

$$\mathbb{C}_u = \{z \in \mathbb{C} | \Im(z) > 0\}, \qquad \mathbb{C}_l = \{z \in \mathbb{C} | \Im(z) < 0\},$$

denote the upper and lower complex half-planes, and let

$$u(x,y) = u(z) = \Re(f(z)), \qquad v(x,y) = v(z) = \Im(f(z)) \qquad \forall z = (x,y) \in \mathbb{C}.$$

The hypothesis $f^{-1}(\mathbb{R}) \subseteq \mathbb{R}$ implies that

$$\mathbb{C}_u \subseteq f^{-1}(\mathbb{C}_u) \cup f^{-1}(\mathbb{C}_l), \qquad \mathbb{C}_l \subseteq f^{-1}(\mathbb{C}_u) \cup f^{-1}(\mathbb{C}_l).$$

Since f is continuous, $f^{-1}(\mathbb{C}_u)$ and $f^{-1}(\mathbb{C}_l)$ are open. If $f^{-1}(\mathbb{C}_u)$ was empty, then $\frac{1}{f-i}$ would be a bounded entire function, and therefore by Liouville's theorem f would be a constant. Likewise, if $f^{-1}(\mathbb{C}_l)$ was empty, by Liouville's theorem applied to $\frac{1}{f+i}$, f would be a constant. Thus, $f^{-1}(\mathbb{C}_u)$ and $f^{-1}(\mathbb{C}_l)$ are nonempty. So either

$$\begin{aligned}
&\mathbb{C}_u \subseteq f^{-1}(\mathbb{C}_u) \text{ and } \mathbb{C}_l \subseteq f^{-1}(\mathbb{C}_l),\\
&\text{or}\\
&\mathbb{C}_u \subseteq f^{-1}(\mathbb{C}_l) \text{ and } \mathbb{C}_l \subseteq f^{-1}(\mathbb{C}_u),
\end{aligned}$$

because $\mathbb{C}_u$ and $\mathbb{C}_l$ are connected. That is to say, either

$$\left.\begin{array}{llll}
v(x,y) > 0 & \text{and} & v(x,-y) < 0 & \forall x \in \mathbb{R}, y \in]0,\infty[,\\
\text{or} & & & \\
v(x,y) < 0 & \text{and} & v(x,-y) > 0 & \forall x \in \mathbb{R}, y \in]0,\infty[.
\end{array}\right\} \tag{2}$$

Plainly, this implies that

$$v(x,0) = 0 \qquad \forall x \in \mathbb{R}.$$

Thus $v(0,0) = 0$ and $\frac{\partial v}{\partial x}(x,0) = 0$ for all $x \in \mathbb{R}$, and since $f'(x) = \frac{\partial u}{\partial x}(x,0) + i\frac{\partial v}{\partial x}(x,0)$, it follows that $f(0) \in \mathbb{R}$ and $f'(0) \in \mathbb{R}$. Therefore, replacing f by $\frac{1}{f'(0)}(f - f(0))$ if

$f'(0) \neq 0$, or f by $f - f(0)$ if $f'(0) = 0$, the new function satisfies (1); hence, there is no loss in generality to normalize f in such a way that $f(0) = 0$, and $f'(0) = 1$ or 0. If $f'(0) = 1$, then we have

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

and we shall show that $a_2 = a_3 = \cdots = 0$. By (2), if $\Im(z) \neq 0$, then either

$$\Im(z)\ \Im(f(z)) > 0,$$

or

$$\Im(z)\ \Im(f(z)) < 0.$$

We can suppose that $\Im(z)\ \Im(f(z)) > 0$; otherwise we consider $-f$ instead of f. Let $r > 0$. Replacing $f(z)$ by $\frac{1}{r} f(rz)$ gives

$$\Im(z)\ \Im(f(rz)) > 0.$$

Define $\varphi(0) = 1$, and $\varphi(z) = (1 - z^2)\frac{f(rz)}{rz}$ if $z \neq 0$. Then φ is an entire function, and for all z with $|z| = 1$, $z \notin \mathbb{R}$, we have

$$\Re(\varphi(z)) = \frac{2}{r}\Im(z)\ \Im(f(rz)) > 0,$$

and if $z = \pm 1$ then clearly $\Re(\varphi(z)) = 0$. Therefore by the Maximum Principle for harmonic functions, φ maps the unit disc into the right half plane. Let

$$\varphi(z) = 1 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots$$

Then $\psi = \frac{\varphi - 1}{\varphi + 1}$ is a mapping from the unit disc into the unit disc, and $\psi(0) = 0$. Thus by Schwarz's Lemma

$$|\psi'(0)| \leq 1.$$

That is,

$$|d_1| = |\varphi'(0)| \leq 2.$$

For any positive integer n, on an analytic branch of $z^{\frac{1}{n}}$ this can be applied to

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(z^{\frac{1}{n}}e^{\frac{2\pi ik}{n}}) = 1 + d_n z + d_{2n}z^2 + \cdots.$$

So, we obtain $|d_n| \leq 2$. Since

$$\varphi(z) = (1 - z^2)\frac{f(rz)}{rz},$$

and

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,$$

we get

$$d_1 = a_2 r, \qquad d_2 = a_3 r^2 - 1, \qquad d_3 = a_4 r^3 - a_2 r, \qquad \cdots,$$

and since $r > 0$ is arbitrary, the inequality $|d_n| \leq 2$ implies $a_n = 0$ for all $n \geq 2$. Therefore f has the form

$$f(z) = a_1 z + a_0 \qquad \forall z \in \mathbb{C},$$

with $a_0, a_1 \in \mathbb{R}$, $a_0 = f(0)$ and $a_1 = f'(0)$. If $f'(0) = 0$, replacing f by $f - f(0)$ and letting $\varphi(0) = 0$, $\varphi(z) = (1 - z^2)\frac{f(rz)}{rz}$ if $z \neq 0$, as before it follows that φ is an entire function mapping the unit disc into the right half plane. However, since $\varphi(0) = 0$, by the Minimum Principle for harmonic functions φ is a constant, and since f is an entire function it follows that $f \equiv 0$. $\square$

The two cases of $f'(0) = 1$ or 0 in the above proof may be considered simultaneously by letting $\varphi(0) = a_1 \in \mathbb{R}$; however, the computation and the ideas of the proof are made more transparent when treated as above.

Our first proof of the above Lemma was based on the big Picard theorem. The elementary proof presented here was suggested by the referee. The authors are indebted to the referee for this proof, and for the careful review of this article.

Theorem. *Let $L_1, L_2 \subseteq \mathbb{C}$ be two straight lines in $\mathbb{R}^2 \sim \mathbb{C}$. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that*

$$f(z) \in L_2 \quad \Rightarrow \quad z \in L_1 \qquad \forall z \in \mathbb{C}. \tag{3}$$

Then f is a polynomial of the first degree.

Proof. Let $z_1 \in L_1$, $z_2 \in L_2$, and $\theta_1, \theta_2 \in [0, 2\pi[$ be such that θ_i is the angle (in the usual sense) between the x-axis and L_i for $i \in \{1, 2\}$. Then

$$L_1 = \{z_1 + te^{i\theta_1} | t \in \mathbb{R}\}, \qquad L_2 = \{z_2 + te^{i\theta_2} | t \in \mathbb{R}\}. \tag{4}$$

Let

$$g(z) = (f(z_1 + ze^{i\theta_1}) - z_2)e^{-i\theta_2} \qquad \forall z \in \mathbb{C}. \tag{5}$$

Clearly g is an entire function. Let $z \in \mathbb{C}$ be such that $g(z) \in \mathbb{R}$. Then by (5) and (4), we have

$$f(z_1 + ze^{i\theta_1}) = z_2 + g(z)e^{i\theta_2} \in L_2.$$

Hence by (3), we get $(z_1 + ze^{i\theta_1}) \in L_1$. Consequently, by (4), there exists $t \in \mathbb{R}$ such that

$$z_1 + ze^{i\theta_1} = z_1 + te^{i\theta_1}.$$

This implies $z = t \in \mathbb{R}$. Thus

$$g(z) \in \mathbb{R} \quad \Rightarrow \quad z \in \mathbb{R} \qquad \forall z \in \mathbb{C}.$$

Therefore, by the Lemma, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$g(z) = \alpha z + \beta \qquad \forall z \in \mathbb{C}. \tag{6}$$

By extracting f from (5), we get

$$f(w) = z_2 + g((w - z_1)e^{-i\theta_1})e^{i\theta_2} \qquad \forall w \in \mathbb{C}.$$

It follows by (6) that

$$f(w) = z_2 + \{\alpha((w - z_1)e^{-i\theta_1}) + \beta\}e^{i\theta_2}$$

$$= aw + b \qquad \forall w \in \mathbb{C},$$

where

$$a = \alpha e^{i(\theta_2 - \theta_1)} \qquad \text{and} \qquad b = z_2 + (\beta - \alpha z_1 e^{-i\theta_1})e^{i\theta_2}.$$

$\square$

References

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