EMBEDDING FINITE PARTIAL LINEAR SPACES IN FINITE TRANSLATION NETS

G. ERIC MOORHOUSE AND JASON WILLIFORD

Abstract. In 1979 Paul Erdős posed the problem of whether all finite partial linear spaces $\mathbb{L}$ are embeddable in finite projective planes. Except for the case when $\mathbb{L}$ has a unique embedding in a projective plane with few additional points, very little has been done which is directly applicable to this problem. In this paper it is proved that every finite partial linear space $\mathbb{L}$ is embeddable in a finite translation net generated by a partial spread of a vector space of even dimension. The question of whether every finite partial linear space is embedded in a finite André net is also explored. It is shown that for each positive integer n there exist finite partial linear spaces which do not embed in any André net of dimension less than or equal to n over its kernel.

1. Introduction

We define an *incidence system* to be a triple $\mathbb{L} = (\mathcal{P}, \mathcal{L}, I)$ consisting of a set $\mathcal{P}$ of 'points', a set $\mathcal{L}$ of 'lines', and an incidence relation $I \subseteq \mathcal{P} \times \mathcal{L}$. In the cases where the choice of incidence relation is understood we simply write $\mathbb{L} = (\mathcal{P}, \mathcal{L})$. Such an incidence system is a *partial linear space* if

(i) any two distinct blocks meet in at most one point, and
(ii) each block contains at least two points.

A point-line pair (P, ℓ) in $\mathbb{L}$ is called a *flag* or an *antiflag* accordingly as $P \in \ell$ or $P \notin \ell$.

If (i) fails then one typically substitutes the term 'block' for 'line'. We shall be primarily interested in *finite* partial linear spaces (those with only finitely many points and lines). Note that axiom (i) is selfdual (i.e. (i) is equivalent to the statement that any two distinct points lie on at most one common line), while (ii) is not. We shall have no real use for axiom (ii), although we have included it in order to conform to common usage; and it has no bearing on the questions of embeddability that we will consider.

Let $\mathbb{L} = (\mathcal{P}, \mathcal{L})$ and $\widetilde{\mathbb{L}} = (\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$ be two incidence systems. An embedding $\alpha : \mathbb{L} \to \widetilde{\mathbb{L}}$ is a pair of injections

$$\alpha_1 : \mathcal{P} \to \widetilde{\mathcal{P}}, \qquad \alpha_2 : \mathcal{L} \to \widetilde{\mathcal{L}}$$

such that $\alpha_1(P) \in \alpha_2(\ell)$ whenever $P \in \ell$ (for $P \in \mathcal{P}$, $\ell \in \mathcal{L}$). Such an embedding is *strong* if $\alpha_1(P) \in \alpha_2(\ell) \iff P \in \ell$.

In this paper we explore the following question:

Problem 1. *Is every finite partial linear space embedded in a finite projective plane?*

This was posed by Paul Erdős in [4]. He did not include an opinion as to its truth or falsity, only the comment "I have no idea how to attack this problem." It is unclear if the problem was folklore before this time, and Erdős apologizes in advance in [4] if the problem had been formulated before. It is well known that every partial linear space embeds in some projective plane, by a process of free closure due to Hall (see [5]). In particular, every finite partial linear space is embedded in a projective plane with at most countably many points (moreover, there is a projective plane of countable order containing all finite projective planes!). The finite analogue posed by Erdős, however, remains open. Indeed, intuition among finite geometers seems to indicate a high degree of skepticism as to whether Problem 1 is true.

It should be noted that there are finite partial linear spaces which do not embed in any classical plane; the simplest examples are those that violate Desargues' configuration.

One may also ask Problem 1 with respect to strong embeddings. The following Lemma shows these to be equivalent.

Lemma 1. *The following two statements are logically equivalent.*

(a) *Every finite partial linear space embeds in some finite projective plane.*
(b) *Every finite partial linear space strongly embeds in some finite projective plane.*

Proof. Suppose (a) holds and let $\mathbb{L} = (\mathcal{P}, \mathcal{L})$ be a finite partial linear space. For each pair points P, Q which are not collinear, create a new line $\ell_{P,Q} = \{P, Q\}$ through P and Q. Now $\mathbb{L}$ is naturally strongly embedded in $\widetilde{\mathbb{L}} = (\mathcal{P}, \widetilde{\mathcal{L}})$ where $\widetilde{\mathcal{L}} = \mathcal{L} \cup \{\ell_{P,Q} : \{P, Q\} \in \mathcal{A}\}$ where $\mathcal{A}$ is the set of unordered pairs of noncollinear points in $\mathbb{L}$. By (a), there is an embedding $\widetilde{\mathbb{L}} \to \Pi$ of $\widetilde{\mathbb{L}}$ in some finite projective plane Π. It is not hard to see that this embedding and the composite embedding $\mathbb{L} \to \widetilde{\mathbb{L}} \to \Pi$ are strong, so (b) follows. The converse is immediate. $\square$

For this reason we will not require embeddings to be strong embeddings.

To date, there are mainly two types of embedding results which relate to Problem 1: a) results which demonstrate when certain partial linear spaces that are parametrically very close to projective planes can be uniquely completed to projective planes (see [1], [2] for example) and b) studies of partial linear spaces which do not embed in classical projective planes (see [7]). These results, of interest in their own right, seem too restrictive to lead to a solution of Problem 1. We will, however, prove that the local structure of André planes of bounded kernel is restricted, though unlike classical planes our proof is non-constructive.

Concerning Problem 1, given the fact that projective planes exist in great variety and are perhaps hopeless to classify, and that the classical planes have a very restricted substructure, it is difficult to intuit the truth or falsity of Problem 1, and unclear where to begin an investigation. The work in this paper is motivated by the following statement, which we pose as a conjecture.

Conjecture 1. *Every finite partial linear space is embedded in a finite translation plane.*

In this vein we prove that every finite partial linear space is embedded in a finite translation net generated by a partial spread. We also address the question of whether every finite partial linear space is embedded in an André plane.

A few additional definitions are needed before we proceed.

We will denote a vector space of dimension n over the field $\mathbb{F}_q$ by $V(n,q)$. A *partial mixed spread* S of a vector space $V(2n,q)$ is a collection of subspaces of V of dimension no greater than n which intersect trivially (in the origin). If all of the subspaces of S have dimension n then we refer to S as a *partial spread*. Any partial spread of $V(2n,q)$ of size q^n+1 is called a *spread*. A partial linear space $\mathbb{L}$ with point set equal to the vectors of $V(2n,q)$ is said to be *generated by* a partial mixed spread S if the lines of $\mathbb{L}$ are given by the cosets of the members of S. If the partial mixed spread S is a partial spread, we refer to the partial linear space generated by S as a translation net. It is a well-known fact that the translation net generated by a spread is an affine plane commonly referred to as a translation plane. Any affine plane is embeddable in a projective plane by the addition of a line, and consequently the resulting projective plane is finite if and only if the affine plane is finite (for more details on translation planes the reader is referred to [3] and [6]).

We now define a certain class of translation planes, referred to as André planes. Let $F=\mathbb{F}_q$ be a finite field, and let $E=\mathbb{F}_{q^k}$ be an extension of degree k. Let $G=Gal(E/F)$, a cyclic group of order k generated by the automorphism $a\mapsto a^q$. Let $N:E\to F$ be the norm map $a\mapsto a^{1+q+q^2+\cdots+q^{k-1}}$. Denote by $F^\times$ the set of nonzero elements of F, and let $\phi:F^\times\to G$ be any map satisfying $\phi(1)=1$. Define the binary operation '$*$' on E by

$$x*m=\begin{cases} x^{\phi(N(m))}m, & \text{if } m\neq 0;\\ 0, & \text{if } m=0.\end{cases}$$

The algebraic structure $(E,+,*)$ is called an André quasifield, and the following lines form an affine plane called an André plane.

(1) $\{a\}\times E$ for $a\in E$ (the line "$x=a$"); and
(2) $\{(x,x*m+b):x\in E\}$ for $m,b\in E$

André planes form a particular class of translation planes, though we will not make use of their corresponding representation as spreads in vector spaces. A given André plane is said to have *kernel* F if F is the largest field such that the given plane may be constructed in the manner described above.

Let $\mathcal{A}(q,k)$ denote the set of $(q-2)^k$ such affine planes (corresponding to the $(q-2)^k$ possible choices of ϕ). This may be described as the class of André planes of order q^k with kernel containing F (hence of dimension at most k over the kernel). We will also make use of a point-block incidence structure, defined below. With q,k,E,F,G as above, we define $C(q,k)$ to be the incidence structure having point set E^2 and blocks

(1) $\{a\}\times E$ for $a\in E$ (the line "$x=a$"); and
(2) $\{(x,x^\sigma m+b):x\in E\}$ for $m,b\in E$ and $\sigma\in G$ (the line "$y=x^\sigma m+b$").

In general this structure is not a partial linear space; but clearly every member of $\mathcal{A}(q,k)$ embeds in $C(q,k)$.

2. Main Results

Our first aim is to prove that every finite partial linear space $\mathbb{L}$ is embedded in a translation net generated by a partial spread.

Theorem 1. *Let* $\mathbb{L} = (\mathcal{P}, \mathcal{L})$ *be a finite partial linear space,* p *a prime and* n *an integer satisfying* $n \geq max\{\frac{|\mathcal{P}|}{2}\} \cup \{|\ell| : \ell \in \mathcal{L}\}$ *and* t *an integer such that* $p^t \geq |\mathcal{L}| - 1$*. Let* $t = t_1 t_2$ *be any factorization of* t *into two positive integers. Then* $\mathbb{L}$ *is embedded in a finite translation net generated by a partial spread* S *in* $V(2nt_1, p^{t_2})$*. Furthermore, we can take* S *to satisfy* $|S| \geq q^n + 1 - (|\mathcal{L}| - 3)\sum_{i=0}^{n-1} q^i$*, where* $q = p^{t_2}$*.*

To prove this, we will first embed $\mathbb{L}$ in a partial linear space generated by a partial mixed spread.

Theorem 2. *Let* $\mathbb{L} = (\mathcal{P}, L)$ *be a finite partial linear space. Let* q *be a prime power and* V *be a vector space of dimension equal to* $2n$ *where* $n \geq max\{\frac{|\mathcal{P}|}{2}\} \cup \{|\ell| : \ell \in \mathcal{L}\}$*. Let* $B = \{e_1, \ldots, e_{2n}\}$ *be a basis of* V*. Let* $\phi : \mathcal{P} \to B$ *be any injective map. Define for* $\ell \in L$ *the set* $\phi(\ell) = \{\sum_{P \in \ell} \lambda_P \phi(P) : \sum_{P \in \ell} \lambda_P = 0\}$*. Then* $S = \{\phi(\ell) : \ell \in \mathcal{L}\}$ *is a partial mixed spread of* V*. Furthermore,* $\mathbb{L}$ *is embedded in the partial linear space generated by* S*.*

We then proceed to extend this partial mixed spread to a partial spread.

Theorem 3. *Let* q *be a prime power and* S *a partial mixed spread of* $V(2n, q)$ *with* $|S| - 1 \leq q$*. Then there exists a partial spread* $\widetilde{S}$ *of* V *such that every element of* S *is a subspace of some element of* $\widetilde{S}$*, with no two elements of* S *being subspaces of the same element of* $\widetilde{S}$*.*

We then include a result which may be viewed as the translation plane analogue of the free-plane construction.

Theorem 4. *Let* $\mathbb{L}$ *be a finite partial linear space and* $F_1 \subset F_2 \subset \ldots$ *be a chain of finite field extensions with* $F_i \neq F_{i+1}$ *for all* i*, and let* $F = \cup F_i$*. Then* $\mathbb{L}$ *is embedded in a translation plane of finite dimension over* F*. In particular,* $\mathbb{L}$ *is embedded in a translation plane of finite dimension over the algebraic closure of any finite field.*

We then investigate embeddings in André planes. We remind the reader that $A(q, k)$ is the set of all André planes of order q^k containing $\mathbb{F}_q$ in their kernel, and that $C(q, k)$ is an incidence structure which contains an embedded copy of every member of $A(q, k)$.

Theorem 5. *For every* k *there exists a partial linear space which does not embed in any* $C(q, t)$ *for* $t \leq k$*, regardless of the choice of* q*.*

This gives us an immediate corollary about André nets.

Corollary 1. *For every* k *there exists a partial linear space which does not embed in any member of* $\mathcal{A}(q, t)$ *for* $t \leq k$*, regardless of the choice of* q*.*

This leads us to the conclusion that arbitrarily large dimensions are necessary if one is to obtain existence results for embeddings using André planes.

Given the nature of Theorem 5 it is natural to ask whether every partial linear space is embeddable into $C(q, k)$ for sufficiently large k. In actuality, it happens that any finite point block incidence structure is strongly embedded in $C(q, k)$. Denote by $\Pi(n)$ the product of the first n primes, so that for example $\Pi(3) = 2{\cdot}3{\cdot}5 = 30$.

Theorem 6. *Let* $\mathbb{L}$ *be a finite point-block incidence system with* m *points and* n *blocks. Then* $\mathbb{L}$ *is strongly embedded in* $C(q, k)$ *for some* $k \leq \Pi(n)\lceil \ln(m+1)/\ln q \rceil$*.*

3. Proofs of the Theorems

We begin by proving our second and third theorems, from which Theorem 1 will follow.

Proof of Theorem 2

Let $\mathbb{L} = (\mathcal{P}, \mathcal{L})$, q, n, B and ϕ satisfy the conditions of the theorem. Let $\ell, \widetilde{\ell} \in \mathcal{L}$. If $\phi(\ell), \phi(\widetilde{\ell})$ meet in a point R then $R = \sum_{P \in l} \lambda_P \phi(P) = \sum_{Q \in \widetilde{\ell}} \lambda_Q \phi(Q)$. As ℓ and $\widetilde{\ell}$ meet in at most one point, at most one of the coefficients λ_P is nonzero. However, $\sum_{P \in \ell} \lambda_P = 0$ and therefore all the coefficients λ_P are zero, and R is the origin. Let $S = \{\phi(\ell) : \ell \in \mathcal{L}\}$. Since the dimension of each element of S is at most n, S is a partial mixed spread of V. We now verify that $\mathbb{L}$ is embedded in the partial linear space generated by S. For each line ℓ, we choose a point P_ℓ such that $P_\ell \in \ell$. Define $t_\ell = \phi(\ell) + \phi(P_\ell)$ and let $T = \{t_\ell : \ell \in L\}$. Note that $\phi(P_\ell) \in t_\ell$ and if Q is any other point of a line ℓ, note that $\phi(Q) - \phi(P_\ell) \in \phi(\ell)$, so $\phi(Q) \in t_\ell$. Therefore, the map $\psi : \mathbb{L} \to T$ defined by $\psi : P \mapsto \phi(P)$ and $\psi : \ell \mapsto t_\ell$ gives the desired embedding. □

Proof of Theorem 3

Let q be a prime power, and let S be a partial mixed spread of a vector space V of dimension $2n$ satisfying $|S| - 1 \leq q$. If S is already a partial spread we are done. If not, let $s \in S$ be such that $d < n$, where d is the dimension of s. We then show the existence of a $d+1$-dimensional subspace of V which contains s but does not intersect with any of the other members of S. Let T be the set of all $d+1$-dimensional subspaces of V which contain s. Since members of T correspond to 1-spaces of V/S, we have $|T| = \sum_{i=0}^{2n-d-1} q^i$. By the dimension theorem, no member of T can intersect any member of $S \backslash \{s\}$ in more than a 1-dimensional subspace, and every two distinct members of T must intersect precisely in s. Since each member of S has dimension at most n, we have that the total number of members of T which intersect $S \backslash \{s\}$ in a 1-dimensional subspace is at most $(|S| - 1) \sum_{i=0}^{n-1} q^i$. Since $(|S| - 1) \leq q$ and $d < n$, we have $\sum_{i=0}^{2n-d-1} q^i - (|S|-1) \sum_{i=0}^{n-1} q^i > 0$, so there exists $\widetilde{s}$ of dimension $d+1$ containing s and with $\widetilde{S} = (S \backslash \{s\}) \cup \{\widetilde{s}\}$ a new partial mixed spread of V. We may repeat this process as long as there is a subspace which has dimension less than n, resulting in the desired partial spread of V. □

We now are ready to prove Theorem 1.

Proof of Theorem 1

Let $\mathbb{L} = (\mathcal{P}, \mathcal{L})$ be a finite partial linear space, p a prime and n an integer satisfying $n \geq max\{\frac{|\mathcal{P}|}{2}\} \cup \{|\ell| : \ell \in \mathcal{L}\}$ and t an integer such that $p^t \geq |\mathcal{L}| - 1$. Let V be a vector space of dimension $2n$ ever $\mathbb{F}_{p^t}$, and W be a vector space of dimension $2nt_1$ over $\mathbb{F}_{p^{t_2}}$. Since $p^t \geq |\mathcal{L}| - 1$, by 3 there exists a partial spread S of V that generates a translation net containing an embedded copy of $\mathbb{L}$. Let $t = t_1 t_2$ be any factorization of t into two positive integers. Since $\mathbb{F}_{p^t}$ is a vector space of dimension t_1 over $\mathbb{F}_{p^{t_2}}$, there is a natural (though not unique) way to identify vectors of V with vectors of W. Since the resulting vectors are closed under scalar multiplication from $\mathbb{F}_{p^{t_2}}$, this identification maps subspaces of V to subspaces of

W. The image of S is then a partial spread of W. $\square$

We now give a proof of Theorem 4.

Proof of Theorem 4

Let $\mathbb{L} = (\mathcal{P}, \mathcal{L})$ be a finite partial linear space. Let $F_1 \subset F_2 \subset \dots$ be a chain of finite field extensions with $F_i \neq F_{i+1}$ for all i, and let $F = \cup F_i$. Let k be a positive integer such that $|F_k| > |\mathcal{L}| - 1$. Using the previous results, we can find an integer n and partial spread S of $V(2n, F_k)$ such that $\mathbb{L}$ is embedded in the translation net generated by S. We give an algorithm whose limiting structure is the desired spread of $V(2n, F)$. Given a partial mixed spread T of $V(2n, F_i)$ for some i:

(i) If T is not a partial spread, we find a field extension F_j such that $|T| - 1 \leq |F_j|$. Using Theorem 1 we construct a partial spread $\widetilde{T}$ of $V(2n, F_j)$ whose members contain the members of T. Go to step (ii).

(ii) If T is a partial spread, let v be a vector of $V(2n, F_i)$ which is in no member of T and which is minimal with respect to the size of the field generated by its components. Let t be the one dimensional subspace generated by v, and $\widetilde{T} = T \cup \{t\}$. Go to step (i).

Clearly any given vector of $V(2n, F)$ will eventually be contained in a subspace of dimension n. $\square$

We now turn our attention to the theorems concerning André planes. Before proving 5, we will require the following Lemma.

Lemma 2. *Let $k \geq 1$ and let $\mathbb{L}$ be a partial linear space. Then there exists a partial linear space $\widetilde{\mathbb{L}}$ such that for every k-coloring of the lines of $\mathbb{L}$, there is an embedding $\alpha : \mathbb{L} \to \widetilde{\mathbb{L}}$ such that the lines of $\alpha(\mathbb{L}) \subseteq \widetilde{\mathbb{L}}$ all have the same color.*

Proof of Lemma 2:

Consider the *incidence graph* $\Gamma_{\mathbb{L}}$ of $\mathbb{L} = (\mathcal{P}, \mathcal{L})$: this is the bipartite graph with vertex set $\mathcal{P} \cup \mathcal{L}$, and whose edges correspond to the flags of $\mathbb{L}$. Since $\mathcal{L}$ is a partial linear space, $\Gamma_{\mathbb{L}}$ has no cycle of length 4. By a result of Nešetřil and Rödl (see [8], Theorem 6.3), there exists a bipartite graph $\widetilde{\Gamma}$ with no 4-cycle, such that for every k-coloring of the edges of $\widetilde{\Gamma}$, there is a monochromatic subgraph isomorphic to $\Gamma_{\mathbb{L}}$. Now we may view $\widetilde{\Gamma}$ as the bipartite incidence graph $\Gamma_{\widetilde{\mathbb{L}}}$ of some partial linear space $\widetilde{\mathbb{L}}$. Consider any k-coloring $\gamma : \widetilde{\mathcal{L}} \to \{1, 2, \dots, k\}$ of the lines of $\widetilde{\mathbb{L}} = (\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$. This induces a k-coloring of the flags of $\widetilde{\mathbb{L}}$ (i.e. edges of $\widetilde{\Gamma}$): simply take $\gamma(P, \ell) = \gamma(\ell) \in \{1, 2, \dots, k\}$ for every flag (P, ℓ) of $\widetilde{\mathbb{L}}$. Now there is an embedding $\alpha : \mathbb{L} \to \widetilde{\mathbb{L}}$ such that all flags of $\alpha(\mathbb{L}) \subseteq \widetilde{\mathbb{L}}$ have the same color $\in \{1, 2, \dots, k\}$. Then every line $\alpha(\ell) \in \alpha(\mathcal{L})$ has this same color. $\square$

We are now ready to prove Theorem 5.

Proof of Theorem 5:

Let $\mathbb{L}$ be a partial linear space which does not embed in any Desarguesian plane (for example, a partial linear space violating Desargues Theorem). Let $\widetilde{\mathbb{L}} = (\widetilde{\mathcal{P}}, \widetilde{\mathcal{L}})$

be as in the Lemma. Suppose $\beta : \widetilde{\mathbb{L}} \to C(q,k)$ is an embedding, so that β consists of a pair of embeddings

$$\beta_1 : \widetilde{\mathcal{P}} \to E^2,$$
$$\beta_2 : \widetilde{\mathcal{L}} \to \{\text{blocks of } C(q,k)\}.$$

Consider the coloring $\gamma : \widetilde{\mathcal{L}} \to G$ defined as follows: $\gamma(\ell) = \sigma$ whenever $\beta_2(\ell)$ is a block of the form $y = x^\sigma m + b$; and we set $\gamma(\ell) = 1$ whenever $\beta_2(\ell)$ is a block of the form $x = a$. By Lemma 2, there exists an embedding $\alpha : \mathbb{L} \to \widetilde{\mathbb{L}}$ such that under the composite embedding $\beta \circ \alpha : \mathbb{L} \to C(q,k)$, all blocks have the same $\sigma \in G$. Now consider the isomorphism $\psi : C(q,k) \to C(q,k)$ defined by $(x,y) \mapsto (x^{\sigma^{-1}}, y)$. Then ψ maps all blocks of $\beta(\alpha(\mathbb{L})) \subseteq C(q,k)$ to blocks of the form $y = xm + b$ or $x = a$. Thus $\psi \circ \beta \circ \alpha$ is an embedding from $\mathbb{L}$ into the Desarguesian affine plane over E, a contradiction. $\square$

We now turn our attention to Theorem 6.

It is convenient to represent an arbitrary point-block incidence system (with m points and n blocks, say) by its incidence matrix. Thus we take $A = \big(a_{ij}\big)$ to be an arbitrary $m \times n$ matrix of 0's and 1's. [*Aside:* The corresponding incidence system is a partial linear space iff $A^T A$ has no off-diagonal entry exceeding 1; and diagonal entries *must* exceed 1, if we require every line to have at least two points.] Let $F = \mathbb{F}_q$, and let $\overline{F}$ be the algebraic closure of F. By a *realization of A over F,* we mean a tuple $(c_1, c_2, \ldots, c_m\,;\, \sigma_1, \sigma_2, \ldots, \sigma_n)$ where $c_i \in \overline{F}$ and $\sigma_j \in G(\overline{F}/F)$, such that $c_i^{\sigma_j} = c_i$ iff $a_{ij} = 1$. We show that such realizations always exist; and that for fixed q, such realizations allow us to embed an arbitrary finite incidence structure in $C(q,k)$ for some k.

Lemma 3. *Let $p_1, p_2, \ldots, p_n$ be distinct primes, and let $\alpha_i \in \overline{\mathbb{F}_q}$ be a primitive $(q^{p_i}-1)$-th root of unity. Then $\mathbb{F}_q(\alpha_1\alpha_2\cdots\alpha_n) = \mathbb{F}_{q^N}$ where $N = p_1 p_2 \cdots p_n$.*

Proof. Clearly we may assume $n \geq 2$. Let $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \mathbb{F}_{q^N}$ and suppose that $\mathbb{F}_q(\alpha) \subset \mathbb{F}_{q^N}$ is a proper subfield. Then $\alpha \in \mathbb{F}_{q^{N/p_i}}$ for some $i \in \{1, 2, \ldots, n\}$, and we may suppose that $i = 1$. Since $\alpha_2\alpha_3\cdots\alpha_n \in \mathbb{F}_{q^{p_2 p_3 \cdots p_n}}$ we obtain

$$\alpha_1 \in \mathbb{F}_{q^{p_2 p_3 \cdots p_n}} \cap \mathbb{F}_{q^{p_1}} = \mathbb{F}_q\,,$$

a contradiction. $\square$

Recall that $\Pi(n)$ the product of the first n primes. The Prime Number Theorem is equivalent to the statement that $\ln \Pi(n) \sim n$ as $n \to \infty$; and since the j-th prime is at most 2^j by Bertrand's Postulate, we have the strict upper bound $\ln \Pi(n) \leq \frac{\ln 2}{2} n(n+1)$ for all $n \geq 1$; see [NZM, pp.366–367] for details.

Lemma 4. *For every $m \times n$ matrix A of 0's and 1's and every finite field F, there exists a realization $(c_1, \ldots, c_m\,;\, \sigma_1, \ldots, \sigma_n)$ of A over F. Moreover we may assume that $c_1, \ldots, c_m$ lie in an extension $E \supseteq F$ of degree $[E : F] \leq \Pi(n)$; and that the automorphisms $\overline{\sigma_1}, \ldots, \overline{\sigma_n} \in G(E/F)$ induced by $\sigma_1, \ldots, \sigma_n$ are distinct.*

Proof. Let $p_1 < p_2 < \cdots < p_n$ be the first n primes, and let $N = \Pi(n) = p_1 p_2 \cdots p_n$. For each $i = 1, 2, \ldots, n$, let $\alpha_i \in \overline{F}$ be a primitive $(q^{p_i}-1)$-th root of unity. Let

$\sigma_1,\ldots,\sigma_n \in G(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ such that σ_j has fixed field $\mathbb{F}_{q^{N/p_j}}$; for example we may take $\sigma_j : x \mapsto x^{q^{N/p_j}}$.

Now simply take $c_i = \prod_{j=1}^n \alpha_j^{1-a_{ij}}$. By Lemma 3 we see that $c_i^{\sigma_j} = c_i$ iff $a_{ij} = 1$. □

We are now ready to prove Theorem 6.

Proof of Theorem 6 Let $F = \mathbb{F}_{q^r}$ where $r = \lceil \ln(m+1)/\ln q \rceil$. This ensures that the number of nonzero elements in F satisfies $q^r - 1 \geq m$. Also let A be the point-line incidence matrix of $\mathbb{L}$. By Lemma 4, there exists a realization $(c_1,\ldots,c_m\,;\,\sigma_1,\ldots,\sigma_n)$ of A over F such that $c_1,\ldots,c_m$ lie in an extension $E \supseteq F$ of degree at most $\Pi(n)$; moreover the automorphisms $\overline{\sigma_1},\ldots,\overline{\sigma_n} \in G(E/F)$ induced by $\sigma_1,\ldots,\sigma_n \in G(\overline{F}/F)$ are distinct. We may also assume that the elements $c_1,\ldots,c_m \in E$ are distinct; otherwise replace c_i by $\gamma_i c_i$ where $\gamma_1,\ldots,\gamma_m \in F$ are nonzero scalars chosen so that $\gamma_1 c_1,\ldots,\gamma_m c_m$ are distinct. Now $\mathbb{L}$ strongly embeds in $C(q,k)$ where $k = [E:F] \leq \Pi(n)$ as follows: points are realized by $(c_i,c_i) \in E^2$ for $i = 1,2,\ldots,m$; and blocks are realized by the subsets $y = x^{\overline{\sigma_j}}$ (i.e. $\{(x,x^{\sigma_j}) : x \in E\}$) for $j = 1,2,\ldots,m$. □

4. Concluding Remarks

Suppose that the answer to Problem 1 is 'no' and there is a finite partial linear space $\mathbb{L}$ which does not embed in any finite projective plane. What consequences would this have? Theorem 2 allows us to embed $\mathbb{L}$ into a mixed translation net generated by a partial mixed spread S in some space $V(2n,q)$. While Theorem 1 shows that we may embed this further into a translation net generated by a partial spread with asymptotically q^n lines, it must be the case that we cannot complete this to a spread. Such partial spreads are called maximal partial spreads, and they are well-studied (see [3]). However, we can embed this partial mixed spread into a larger dimensional vector space by adjoining zeros to the end of all of the vectors in the subspaces in S. As the dimension grows, the number of inequivalent spreads increases dramatically (at least exponentially), however none of these spreads can contain the elements of S in separate spread elements. The dimensions of the elements of S are fixed, however, while the dimension of the space is arbitrarily large. For example, imagine a set of twenty 17-dimensional subspaces of a 1000-dimensional vector space V over $\mathbb{F}_2$ which are contained in no spread of V. Then imagine increasing the dimension to a million, then a billion, etc. with all the spreads of each vector space conspiring carefully so as not to contain the twenty seventeen-dimensional members of S. Based on the intuition that this scenario cannot occur, we pose the following stronger conjecture:

Conjecture 2. *For every prime p and partial linear space $\mathbb{L}$, there exists an integer t such that $\mathbb{L}$ is embedded in a translation plane of order q^t.*

The construction of the partial spread of Theorem 1 requires that the dimension $2n$ be sufficiently large. It is not clear if this restriction is necessary in general. The following problem is still open:

Problem 2. *Determine whether there is a finite partial linear space that is not embedded in any translation net generated by a partial spread of* $V(4,q)$*, regardless of the choice of* q*.*

Since spreads are difficult objects to construct and classify, it is unclear if a constructive proof of Conjecture 1 can be given using Theorem 1, particularly if arbitrarily large dimensions are needed. However, non-constructive techniques to attack this problem are so far lacking. In fact, to the authors' knowledge, the only proofs that spreads exist in even dimensional vector spaces are constructive. Non-constructive proofs of this well-known fact may give new techniques which could be used to attack problem 1.

On the other hand, if André planes suffice for the embedding of all finite partial linear spaces, a constructive proof may be possible. The following is a weaker form of the question concerning embeddings in André planes.

Problem 3. *Prove that every finite partial linear space is embedded in a net generated by a partial spread whose components are defined by equations of the form* $y = mx^{\rho} + b$ *where* ρ *is a field automorphism.*

Lastly, we pose the following problem:

Problem 4. *Find a smallest partial linear space which does not embed in any André plane of dimension two over its kernel.*

The answer to this question may shed some light on Problem 2.

References

[1] A. Beutelspacher, K. Metsch, Embedding finite linear spaces in projective planes. *North-Holland Math. Stud.* 123, North-Holland, Amsterdam, 1986.

[2] A. Beutelspacher, K. Metsch, Embedding finite linear spaces in projective planes. II. *Discrete Math.* 66 (1987), no. 3, 219–230.

[3] M. Biliotti, V. Jha, N. L. Johnson, *Foundations of Translation Planes*, Marcel Dekker, INC. New york, Basel (2001).

[4] P. Erdős, Some old and new problems in various branches of combinatorics. *Proceedings of the Tenth Southeastern Conference on Combinatorics, Graph Theory and Computing* Florida Atlantic Univ., Boca Raton, Fla. (1979), pp. 19–37.

[5] M. Hall, Projective planes. *Trans. Amer. Math. Soc.* 54 (1943), 229–277.

[6] D. Hughes, F. Piper, *Projective planes.* Graduate Texts in Mathematics Vol. 6. Springer-Verlag, New York-Berlin, 1973.

[7] R. Lauffer, Die nichtkonstruierbare Konfiguration (10_3). (German) *Math. Nachr.* 11 (1954), 303–304.

[8] J. Nešetřil and V. Rödl, Strong Ramsey theorems for Steiner systems. *Trans. Amer. Math. Soc.* 303 no. 1 (1987), 183–192.

[9] I. Niven, H. S. Zuckerman, H. L. Montgomery, *An Introduction to the Theory of Numbers*, John Wiley and Sons, Inc. (1991).

(G. Eric Moorhouse) Department of Mathematics, University of Wyoming, Laramie, WY 82071

(Jason Williford) Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609-2280