On the Chromatic Numbers of Planes

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Abstract. Define two points of the Euclidean plane $\mathbb{R}^2$ to be *adjacent* if they are at distance 1 from each other. It is known that the chromatic number of the resulting graph satisfies $4 \leqslant \chi(\mathbb{R}^2) \leqslant 7$. We obtain some partial results concerning $\chi(K^2)$ for more general choices of field K. In particular $\chi(\mathbb{R}^2) = \chi(K^2)$ for some number field K, i.e. $\mathbb{Q} \subseteq K \subset \mathbb{R}$ with $[K : \mathbb{Q}] < \infty$; moreover points of K^2 are constructible by straightedge and compass from points of K_0^2 where $K_0 \supseteq \mathbb{Q}$ is a finite extension of odd degree and $\chi(K_0^2) = 2$. In the course of studying $\chi(K^2)$ where K is the real field $\mathbb{R}$ or a number field, we are naturally forced to consider the case where K is a finite field. Finally, we pose the problem of deciding whether or not $\chi(\mathbb{C}^2)$ is finite; and similarly $\chi(K^2)$ for a subfield $K \subseteq \mathbb{C}$ containing i.

1. A Field-Theoretic Approach to the Problem

Let K be any field (or more generally, any commutative ring with unity). We regard K^2 as a graph in which two vertices $(a, b), (c, d) \in K^2$ are *adjacent* iff

$$(1.1)\qquad (a-c)^2 + (b-d)^2 = 1.$$

A natural (and in general very difficult) problem is to determine the chromatic number $\chi(K^2)$ of this graph. We recall first the relevant definitions: A *proper colouring* of K^2 is a map $\psi : K^2 \to \mathcal{C}$ (for some set $\mathcal{C}$) such that $\psi(a, b) \neq \psi(c, d)$ for all $(a, b), (c, d) \in K^2$ satisfying (1.1). The *chromatic number of* K^2, denoted $\chi(K^2)$, is the minimum possible $|\mathcal{C}|$ for which there exists a proper colouring $\psi : K^2 \to \mathcal{C}$. The best that is known concerning the chromatic number of $\mathbb{R}^2$ is that $4 \leqslant \chi(\mathbb{R}^2) \leqslant 7$; see [4,pp.177-180], [6,pp.150–152]. We hope to shed fresh light on this open problem. Guided by a suspicion that $\chi(\mathbb{R}^2)$ is actually equal to 7, our hope is to identify finite subgraphs $\Gamma \subset \mathbb{R}^2$ with as few vertices as possible, and but without any proper 4-colouring. It seems that such subgraphs must have hundreds, if not thousands, of vertices; and Γ must be chosen wisely if available

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computational resources are to have any hope of showing that $\chi(\Gamma) > 4$. Our results are intended to focus the search for good candidates for Γ.

We find that the problem of determining $\chi(\mathbb{R}^2)$ is related to a determination of $\chi(K^2)$ for other rings K, and some results concerning such values $\chi(K^2)$ are presented. In particular the values of $\chi(F^2)$, for finite fields F, play a rôle in this investigation (Section 6). In Section 10 we pose the apparently open question of whether $\chi(\mathbb{C}^2)$ is finite, although this does not bear directly on our investigation of $\chi(\mathbb{R}^2)$.

By the preceding remarks, observe that $2 \leqslant \chi(K^2) \leqslant \chi(L^2) \leqslant 7$ for all subfields $K \subseteq L \subseteq \mathbb{R}$. By a theorem of de Bruijn and Erdős, $\chi(\mathbb{R}^2)$ is the maximum of $\chi(\Gamma)$ among all finite induced subgraphs $\Gamma \subset \mathbb{R}^2$. Since every such finite subgraph Γ has coordinates in a subfield $K \subset \mathbb{R}$ which is finitely generated over $\mathbb{Q}$, we see that $\chi(\mathbb{R}^2)$ is the maximum of $\chi(K^2)$ among all finitely generated subfields $K \subset \mathbb{R}$. In fact this maximum is attained for some number field K, thus:

1.2 Theorem. *There exist subfields $K, K_0 \subset \mathbb{R}$ such that*

(i) $\mathbb{Q} \subseteq K_0 \subset K \subset \mathbb{R}$ *with* $[K:\mathbb{Q}] < \infty$*;*

(ii) $\chi(K^2) = \chi(\mathbb{R}^2)$*;*

(iii) $[K_0:\mathbb{Q}]$ *is odd and* $\chi(K_0^2) = \chi(\mathbb{Q}^2) = 2$*;*

(iv) $[K:K_0] = 2^n$ *for some* $n \geqslant 1$ *and points of* K^2 *are constructible by straightedge and compass from points of* K_0^2*; and*

(v) the normal closure $\widehat{K}$ *of* K *in* $\mathbb{C}$ *satisfies* $\widehat{K} \cap \mathbb{R} = K$*.*

Here we prove Theorem 1.2, postponing two key points until Sections 5 and 7. The fact that $\chi(\mathbb{R}^2) = \chi(K^2)$ for some subfield $K \subset \mathbb{R}$ with $[K:\mathbb{Q}] < \infty$ is shown in Section 5. Let K be such a field; we show how the remaining conclusions of Theorem 1.2 follow. Let $\widehat{K}$ be the closure of K in $\mathbb{C}$. Then $\widehat{K} \cap \mathbb{R} \supseteq K$, and we may assume that equality holds; otherwise replace K by $\widehat{K} \cap \mathbb{R}$ to obtain a new extension K for which (ii) and (v), and the finiteness condition $[K:\mathbb{Q}] < \infty$, all hold.

Now there exist subfields

$$K = K_n \supset K_{n-1} \supset \cdots \supset K_2 \supset K_1 \supset K_0 \supseteq \mathbb{Q}$$

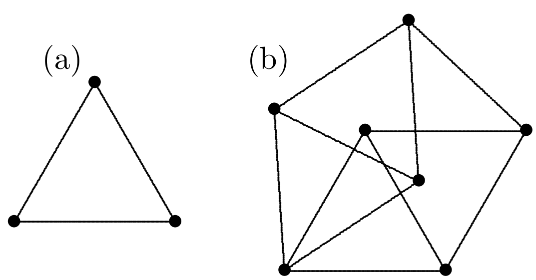
where $[K_j : K_{j-1}] = 2$ for $j = 1, 2, \ldots, n,\ n \geqslant 1$; and $[K_0:\mathbb{Q}]$ is odd. To see this, let $\tau \in G := Gal(\widehat{K}/\mathbb{Q})$ be the Galois automorphism induced by complex conjugation, so that K is the fixed field of τ. We have subgroups $G \geqslant P_0 > P_1 > P_2 > \cdots > P_n = \langle\tau\rangle \geqslant 1$ where

P_0 is a Sylow 2-subgroup of G and $[P_{j-1} : P_j] = 2$ for $j = 1, 2, \ldots, n$. The corresponding fixed fields give the desired tower of fields. Conclusion (iii) follows from Theorem 7.1.

The fact that $[K_j : K_{j-1}] = 2$ means that points in K^2 are constructible by straight-edge and compass, or by compass alone, from the points in K_0^2. So it remains only to check by how much the chromatic number can increase in the case of the quadratic extensions $K_j \supset K_{j-1}$.

Two important configurations in the plane which are compass-constructible from $\mathbb{Q}^2$ are the equilateral triangle of side length one, and the Moser spindle (see Figure 1.3, in which adjacent vertices are indicated by line segments of length one). The smallest number field K for which K^2 contains a 3-cycle, is given by $K = \mathbb{Q}(\sqrt{3})$; see Proposition 1.4. In this case it is clear that $\chi(K^2) \geqslant 3$, and the equality $\chi(K^2) = 3$ follows from Corollary 8.3. More general results concerning $\chi(K^2)$ for real quadratic extensions $K \supset \mathbb{Q}$ are presented in Section 8.

1.3 Figure.
(a) equilateral triangle
(b) Moser spindle



The smallest number field K for which K^2 contains a Moser spindle of edge length one, is the field $K = \mathbb{Q}(\sqrt{3}, \sqrt{11})$. In this case $\chi(K^2) \geqslant 4$. We have not determined the exact value of $\chi(K^2)$ in this case.

1.4 Proposition. *(a) The graph K^2 contains a 3-cycle iff K contains $1/2$ (i.e. K does not have characteristic 2) and $\sqrt{3}$.*

(b) K^2 contains a Moser spindle iff K contains $1/66$, $\sqrt{3}$ and $\sqrt{11}$.

Proof. If K contains $1/2$ and $\sqrt{3}$ then K^2 contains an equilateral triangle with vertices $(0,0)$, $(1,0)$ and $\frac{1}{2}(1, \sqrt{3})$. Conversely, suppose K^2 contains an equilateral triangle with vertices $v_1, v_2, v_3 \in K^2$. We may suppose that $v_1 = (0,0)$; otherwise translate $K^2 \to K^2$, $v \mapsto v - v_1$. Also we may suppose that $v_2 = (1,0)$; otherwise $v_2 = (a,b)$ with $a, b \in K$, $a^2 + b^2 = 1$ and we may rotate $K^2 \mapsto K^2$, $v \mapsto v\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$. Finally, $v_3 = (c,d)$ must satisfy $c^2 + d^2 = (c-1)^2 + d^2 = 1$, i.e. $c = 1/2$ and $d^2 = 3/4$. This proves (a).

If K contains $1/66$, $\sqrt{3}$ and $\sqrt{11}$ then K^2 contains a Moser spindle with coordinates (reading top-to-bottom and left-to-right in Figure 1.3) $\frac{1}{12}\big(15-\sqrt{33}, 5\sqrt{3}+3\sqrt{11}\big)$, $\frac{1}{12}\big(5-\sqrt{33}, 5\sqrt{3}+\sqrt{11}\big)$, $\frac{1}{2}\big(1,\sqrt{3}\big)$, $\frac{1}{2}\big(3,\sqrt{3}\big)$, $\frac{1}{6}\big(5,\sqrt{11}\big)$, $(0,0)$, $(1,0)$. The converse proceeds as in the proof of (a). $\square$

Examples where K^2 contains no 3-cycle, yet $\chi(K^2) \geqslant 3$ are given in Sections 6 and 8. In Section 8 we also exhibit finite fields K for which K^2 contains no Moser spindle, yet $\chi(K^2) \geqslant 5$.

2. Valuations

In this section we review some number theoretic background needed in later sections. Most of this is standard, but much of this is borrowed from older sources or varied sources with differing terminology and notation. Accordingly, and because many of the intended readers have more expertise in graph theory than in number theory, we provide here a brief review of some of the number theoretic background required in the remainder of the paper. For further details, we refer the reader to [1], [8].

Let K be a field. A *valuation on* K is a map $\big|\ \ \big| : K \to [0,\infty)$ satisfying

(V1) $|\alpha| = 0$ iff $\alpha = 0$;

(V2) $|\alpha\beta| = |\alpha||\beta|$ for all $\alpha, \beta \in K$; and

(V3) $|\alpha+\beta| \leqslant |\alpha| + |\beta|$ for all $\alpha, \beta \in K$.

A valuation $\big|\ \ \big|$ is *non-Archimedean* if it satisfies the following stronger form of (V3):

(V3′) $|\alpha+\beta| \leqslant max\{|\alpha|, |\beta|\}$, and equality holds whenever $|\alpha| \neq |\beta|$.

Two valuations of K are *equivalent* iff they yield the same metric topology on K. Let $\mathfrak{p}$ range over an index set for the inequivalent valuations $\big|\ \ \big|_{\mathfrak{p}}$ of K. The extension field $K_{\mathfrak{p}} \supseteq K$ denotes the completion of K relative to $\big|\ \ \big|_{\mathfrak{p}}$.

Now suppose K is an algebraic number field, i.e. a finite extension of $\mathbb{Q}$, and let $\mathcal{O}$ be the ring of algebraic integers in K. A *fractional ideal* of $\mathcal{O}$ is an additive subgroup of K of the form $c\mathfrak{A}$ for some nonzero ideal $\mathfrak{A} \subseteq \mathcal{O}$ and nonzero $c \in K$. The fractional ideals of $\mathcal{O}$ form a multiplicative group with inverses defined by $\mathfrak{A}^{-1} = \{b \in K : b\mathfrak{A} \in \mathcal{O}\}$. Every fractional ideal admits a unique prime factorization

$$\mathfrak{A} = \prod_{\mathfrak{p}} \mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{A})}$$

where the product ranges over all prime ideals $\mathfrak{p} \subset \mathcal{O}$, and the exponents $\nu_{\mathfrak{p}}(\mathfrak{A}) \in \mathbb{Z}$ are almost all zero. Two fractional ideals $\mathfrak{A}, \mathfrak{B}$ satisfy $\mathfrak{A} \subseteq \mathfrak{B}$ iff $\nu_{\mathfrak{p}}(\mathfrak{A}) \geqslant \nu_{\mathfrak{p}}(\mathfrak{B})$ for all prime ideals $\mathfrak{p} \subset \mathcal{O}$. Moreover $\mathfrak{A} \cap \mathfrak{B} = \prod_{\mathfrak{p}} \mathfrak{p}^{c_{\mathfrak{p}}}$ and $\mathfrak{A} + \mathfrak{B} = \prod_{\mathfrak{p}} \mathfrak{p}^{d_{\mathfrak{p}}}$ where $c_{\mathfrak{p}}$ and $d_{\mathfrak{p}}$ are the maximum and the minimum of $\{\nu_{\mathfrak{p}}(\mathfrak{A}), \nu_{\mathfrak{p}}(\mathfrak{B})\}$ respectively. We also write $\nu_{\mathfrak{p}}(\alpha) = \nu_{\mathfrak{p}}(\alpha\mathcal{O})$ for nonzero $\alpha \in K$, and $\nu_{\mathfrak{p}}(0) = -\infty$. For all $\alpha, \beta \in K$, we have

(i) $\nu_{\mathfrak{p}}(\alpha\beta) = \nu_{\mathfrak{p}}(\alpha) + \nu_{\mathfrak{p}}(\beta)$, and

(ii) $\nu_{\mathfrak{p}}(\alpha+\beta) \geqslant min\{\nu_{\mathfrak{p}}(\alpha), \nu_{\mathfrak{p}}(\beta)\}$, with equality whenever $\nu_{\mathfrak{p}}(\alpha) \neq \nu_{\mathfrak{p}}(\beta)$.

The functions $\nu_{\mathfrak{p}} : K \to \mathbb{Z} \cup \{-\infty\}$ are merely the non-Archimedean valuations of K, written 'additively'; in 'multiplicative' notation, they are the maps $\big|\ \ \big|_{\mathfrak{p}} : K \to [0, \infty)$ defined by

$$\big|\alpha\big|_{\mathfrak{p}} = \begin{cases} (N\mathfrak{p})^{-\nu_{\mathfrak{p}}(\alpha)}, & \text{if } \alpha \neq 0; \\ 0, & \text{if } \alpha = 0 \end{cases}$$

where $N\mathfrak{p} = \big|\mathcal{O}/\mathfrak{p}\big|$ is the order of the residue field $\mathcal{O}/\mathfrak{p}$.

The number field K has finitely many inequivalent Archimedean valuations, given by $\alpha \mapsto |\alpha^\sigma|$ where σ is a field embedding $K \to \mathbb{C}$. The number of such valuations is $r_1 + r_2$ where r_1 is the number of real embeddings $K \to \mathbb{C}$, and r_2 is the number of conjugate pairs of nonreal embeddings; note that $[K : \mathbb{Q}] = r_1 + 2r_2$.

2.1 Proposition. *Let $\mathcal{O}$ be the ring of algebraic integers in an algebraic number field K, and let $\mathfrak{p} \subset \mathcal{O}$ be a prime ideal. Define $R_{\mathfrak{p}} = \{a \in \mathcal{O} : \nu_{\mathfrak{p}}(a) \geqslant 0\}$. Then*

(a) $R_{\mathfrak{p}}$ is the set of all a/b where $a, b \in \mathcal{O}$ and $b \notin \mathfrak{p}$.

(b) For every $k \in \mathbb{Z}$ we have $\mathfrak{p}^k R_{\mathfrak{p}} = \{a \in K : \nu_{\mathfrak{p}}(a) \geqslant k\}$. This is an additive subgroup of K for every k. For every $k \geqslant 0$, it is an ideal of $R_{\mathfrak{p}}$ with quotient $R_{\mathfrak{p}}/\mathfrak{p}^k R_{\mathfrak{p}} \cong \mathcal{O}/\mathfrak{p}^k$.

(c) $R_{\mathfrak{p}}$ is a local ring whose only ideals are those in the infinite chain

$$R_{\mathfrak{p}} \supset \mathfrak{p}R_{\mathfrak{p}} \supset \mathfrak{p}^2 R_{\mathfrak{p}} \supset \mathfrak{p}^3 R_{\mathfrak{p}} \supset \cdots \supset \{0\}.$$

(d) Let $k, \ell \geqslant 0$ and let $A = \mathfrak{p}^{-k} R_{\mathfrak{p}}/\mathfrak{p}^\ell R_{\mathfrak{p}}$. If $\nu_{\mathfrak{p}}(2) \geqslant k$, then the squaring operation $x \mapsto x^2$ gives a well-defined map $A \mapsto A$.

Proof. (a) Let $U = \{a/b : a, b \in \mathcal{O},\ b \notin \mathfrak{p}\}$. Clearly $U \subseteq R_{\mathfrak{p}}$. Conversely, consider a nonzero element $\alpha \in R_{\mathfrak{p}}$, and for each prime ideal $\mathfrak{q} \subset \mathcal{O}$, let $c_{\mathfrak{q}}$ and $d_{\mathfrak{q}}$ be the maximum and minimum (respectively) of $\{0, \nu_{\mathfrak{q}}(\alpha)\}$. Then $\alpha\mathcal{O} = \mathfrak{A}\mathfrak{B}^{-1}$ where $\mathfrak{A} = \prod_{\mathfrak{q}} \mathfrak{q}^{c_{\mathfrak{q}}}$ and $\mathfrak{B} = \prod_{\mathfrak{q}} \mathfrak{q}^{d_{\mathfrak{q}}}$ are ordinary ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathcal{O}$. Since $\nu_{\mathfrak{p}}(\alpha) \geqslant 0$, we have $d_{\mathfrak{p}} = 0$ and so $\mathfrak{B} \not\subseteq \mathfrak{p}$. We may choose $b \in \mathfrak{B} \smallsetminus \mathfrak{p}$ and $a = \alpha b \in \alpha\mathfrak{B} = \mathfrak{A}$.

(b) The first equality follows easily from the definitions. Let $k \geqslant 0$, and consider any element $\frac{a}{b} \in R_{\mathfrak{p}}$ as in (a). Since the ideal $\mathfrak{p} \subset \mathcal{O}$ is maximal, we have $\mathcal{O} = b\mathcal{O} + \mathfrak{p}$ and $\mathcal{O} = \mathcal{O}^k = (b\mathcal{O} + \mathfrak{p})^k \subseteq \mathfrak{p}^k + b\mathcal{O}$. Now

$$\tfrac{a}{b} \in \tfrac{a}{b} \subseteq \tfrac{a}{b}\mathfrak{p}^k + a\mathcal{O} \subseteq \mathfrak{p}^k R_{\mathfrak{p}} + \mathcal{O}$$

and so $R_{\mathfrak{p}} = \mathfrak{p}^k R_{\mathfrak{p}} + \mathcal{O}$. Now

$$R_{\mathfrak{p}}/\mathfrak{p}^k R_{\mathfrak{p}} = (\mathfrak{p}^k R_{\mathfrak{p}} + \mathcal{O})/\mathfrak{p}^k R_{\mathfrak{p}} \cong \mathcal{O}/(\mathcal{O} \cap \mathfrak{p}^k R_{\mathfrak{p}}) = \mathcal{O}/\mathfrak{p}^k.$$

(c) Consider a nonzero ideal $J \subseteq R_{\mathfrak{p}}$, and let k be maximal such that $J \subseteq \mathfrak{p}^k R_{\mathfrak{p}}$. (Since $\cap_{k\geqslant 0}\mathfrak{p}^k R_{\mathfrak{p}} = 0$, such k exists.) Choose $r \in J \smallsetminus \mathfrak{p}^{k+1} R_{\mathfrak{p}}$. For every $s \in \mathfrak{p}^k R_{\mathfrak{p}} \smallsetminus \mathfrak{p}^{k+1} R_{\mathfrak{p}}$ we have $\nu_{\mathfrak{p}}(s/r) = k - k = 0$ so $s \in rR_{\mathfrak{p}} \subset J$, so $J = \mathfrak{p}^k R_{\mathfrak{p}}$.

(d) If $\alpha \in \mathfrak{p}^{-k} R_{\mathfrak{p}}$ and $\beta \in \mathfrak{p}^{\ell} R_{\mathfrak{p}}$, then

$$\nu_{\mathfrak{p}}\big((\alpha+\beta)^2 - \alpha^2\big) = \nu_{\mathfrak{p}}(2\alpha\beta + \beta^2) \geqslant min\{\nu_{\mathfrak{p}}(2) - k + \ell,\, 2\ell\} \geqslant \ell. \qquad \square$$

The following is well-known.

2.2 Lemma. *Every element of K is expressible in the form α/β for some $\alpha, \beta \in \mathcal{O}$, not both of which are contained in $\mathfrak{p}$.* $\square$

We remark that the choice of $\alpha, \beta \in \mathcal{O}$ in Lemma 2.2 depends in general on the choice of $\mathfrak{p}$; one cannot hope to choose $\alpha\mathcal{O}$ and $\beta\mathcal{O}$ to be relatively prime unless $\mathcal{O}$ is a principal ideal domain.

2.3 Lemma. *Every solution of $\xi^2 + \eta^2 = 1$ in K is expressible in the form $(\xi, \eta) = (\alpha/\gamma, \beta/\gamma)$ where $\alpha, \beta, \gamma \in \mathcal{O}$ and at most one of α, β, γ lies in $\mathfrak{p}$.*

Proof. We may assume that $\xi\eta \neq 0$.

First suppose that $\nu_{\mathfrak{p}}(\xi) \geqslant 0$. Then $\nu_{\mathfrak{p}}(\eta) \geqslant 0$ and by Lemma 2.2, we may write $\xi = \alpha/\gamma$, $\eta = \beta/\delta$ for some $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ with $\gamma, \delta \notin \mathfrak{p}$. We may assume that $\gamma = \delta$; otherwise rewrite ξ and η using $\gamma\delta \notin \mathfrak{p}$ as a common denominator. Now α and β are not both in $\mathfrak{p}$, and the result follows.

Otherwise $\nu_{\mathfrak{p}}(\xi) = \nu_{\mathfrak{p}}(\eta) < 0$ and $\xi = \alpha/\gamma$, $\eta = \beta/\delta$ for some $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ with $\nu_{\mathfrak{p}}(\alpha) = \nu_{\mathfrak{p}}(\beta) = 0$ and $\nu_{\mathfrak{p}}(\gamma) = \nu_{\mathfrak{p}}(\delta) > 0$. Again by Lemma 6.1, we have $\gamma/\delta = \gamma'/\delta'$

for some $\gamma', \delta' \in \mathcal{O}$ with $\gamma', \delta' \notin \mathfrak{p}$. Then $\xi = \alpha\delta'/\delta'\gamma$ and $\eta = \beta\gamma'/\delta'\gamma$ where $\alpha\delta', \beta\gamma' \notin \mathfrak{p}$. $\square$

3. Connectedness

While the graph $\mathbb{R}^2$ is clearly connected, in general the graph K^2 fails to be connected. For example, it may be shown that $\mathbb{Q}^2$ has infinitely many connected components, each of which is a translate of R^2 where R is the subring consisting of all a/b such that $a, b \in \mathbb{Z}$ and every prime divisor of b is of the form $4k+1$. This fact, which will not be used in the sequel, is shown by a straight-forward exercise.

3.1 Lemma. *Denote by K_0^2 the connected component of K^2 containing $(0,0)$. Then $\chi(K^2) = \chi(K_0^2)$.*

Proof. By considering the translation group of K^2, it is clear that every connected component of K^2 is isomorphic to K_0^2. $\square$

It follows from Proposition 3.5 that for finitely generated subfields $K \subset \mathbb{R}$, the graph K^2 is *never* connected. However, this fact is not strictly required in later sections of this paper, which consider only the proper colourings of the connected component K_0^2.

We first dispose of the rather trivial case in which K has characteristic 2.

3.2 Proposition. *Suppose K has characteristic 2. Then K_0^2 is a complete bipartite graph. In particular, K^2 is bipartite and $\chi(K^2) = 2$. Moreover, K^2 is disconnected for $|K| > 2$.*

Proof. Let $D = \{(a,a) : a \in K\}$, so that $(0,1) + D = \{(a, a+1) : a \in K\}$. Consider a pair of adjacent points $(a,a), (x,y) \in K^2$, so that $(x+a)^2 + (y+a)^2 = (x+y)^2 = 1$, i.e. $x + y = 1$. It follows that the neighbours of every point $(a,a) \in D$ are precisely the points of $(0,1) + D$; similarly, the neighbours of every point of $(0,1) + D$ are precisely the points of D. Thus $K_0^2 = D \cup ((0,1) + D)$ is bipartite, and is properly contained in K^2 for $|K| > 2$. $\square$

Henceforth we focus attention on the case K has characteristic zero or an odd prime. In this case K has zero or two roots of $X^2 = -1$; if such roots exist, they are primitive fourth roots of unity denoted by $\pm i$.

3.3 Lemma. *Suppose K contains primitive fourth roots of unity. Then K^2 is connected.*

Proof. By hypothesis, K contains $1/2$. For each nonzero $t \in K$, the points $u_t = \frac{1}{2}\big(t + t^{-1}, i(t - t^{-1})\big)$ and $v_t = \frac{1}{2}\big(t + t^{-1}, -i(t - t^{-1})\big)$ are neighbours of $(0,0)$ in K^2. Thus K_0^2 contains the point $w_t = u_t + v_t = (t + t^{-1}, 0)$ and the point

$$w_{(1+i)t} + w_{(1-i)t} - w_t = (t, 0).$$

Similarly $(0,t) \in K_0^2$ for every nonzero $t \in K$. Since K_0^2 is an additive subgroup of K^2, it follows that $K_0^2 = K^2$. $\square$

3.4 Proposition. *If K is a finite field of odd order, then K^2 is connected.*

Proof. Let $|K| = q = p^e$ where p is an odd prime. Denote by S the set of all $t \in K$ such that $(t,0) \in K_0^2$; thus S is an additive subgroup of K. The number of neighbours of $(0,0)$ in K^2 is $q + (-1)^{(q-1)/2}$; see [5,p.93]. Pairs $(a,b), (a,-b)$ of neighbours of $(0,0)$ give rise to points $(2a, 0) = (a,b) + (a,-b) \in K_0^2$, so that $|S| \geqslant (q-1)/2$. Since $|S| = p^r$ for some integer r, it follows that $S = K$. Similarly, K_0^2 contains $(0,t)$ for every $t \in K$, and so $K_0^2 = K^2$. $\square$

The converse of Lemma 3.3 fails in general: many choices of field K (such as $\mathbb{R}$, and many finite fields) do not contain i, yet yield a connected graph K^2. Nevertheless the converse of Lemma 3.3 does hold for number fields:

3.5 Proposition. *Let K be a number field. Then K^2 is connected iff K contains a fourth root of unity.*

Proof. Let $\mathcal{O}$ be the ring of algebraic integers in K. For each prime ideal $\mathfrak{p} \subset \mathcal{O}$ we denote by $K_{\mathfrak{p}}$ the completion of K relative to $|\ \ |_{\mathfrak{p}}$.

Suppose K is not connected. By Lemma 3.3, we see that -1 is not a square in K. By the Global Square Theorem [8,p.182] there exist infinitely many prime ideals $\mathfrak{p} \subset \mathcal{O}$ such that -1 is not a square in $K_{\mathfrak{p}}$, the completion of K at $\mathfrak{p}$. (In order to derive this conclusion from [8,p.182] we have used the fact that K has only finitely many inequivalent Archimedean valuations.) Choose a prime ideal $\mathfrak{p} \subset \mathcal{O}$ not containing 2, such that -1 is not a square in $K_{\mathfrak{p}}$. We observe that -1 is not a square in the residue field $\mathcal{O}/\mathfrak{p}$. For suppose there exists $a \in \mathcal{O}$ such that $f(a) \equiv 0 \mod \mathfrak{p}$, where $f(X) = X^2+1 \in \mathcal{O}[X]$. Then $\nu_{\mathfrak{p}}(f(a)) > 2\nu_{\mathfrak{p}}(f'(a)) = 0$ so by Hensel's Lemma [1,p.49], f has a zero in $K_{\mathfrak{p}}$, contrary to the choice of $\mathfrak{p}$. This verifies our claim that -1 is not a square in the residue field $\mathcal{O}/\mathfrak{p}$.

Now consider any neighbour (α, β) of $(0,0)$ in K^2. By Lemma 2.3, we may write $(\alpha, \beta) = (a/c, b/c)$ for some $a, b, c \in \mathcal{O}$, where at most one of a, b, c is in $\mathfrak{p}$. Clearly $c \notin \mathfrak{p}$, for otherwise a/b gives a zero of $f(X)$ in $\mathcal{O}/\mathfrak{p}$. This shows that every neighbour of $(0,0)$ in K^2 lies in R^2 where $R = \{x \in K : \nu_{\mathfrak{p}}(x) \geqslant 0\}$, a proper subring of K. By induction, it follows that $K_0^2 \subseteq R^2$, a proper subset of K^2. $\square$

4. Graph Homomorphisms

Let Γ and Γ' be graphs (ordinary graphs, undirected with no loops or multiple edges). We write $\phi : \Gamma \to \Gamma'$ if ϕ maps vertices of Γ to vertices of Γ'. Such a map is a *graph homomorphism* if $\phi(x)$ and $\phi(y)$ are adjacent in Γ', whenever x and y are adjacent vertices in Γ. Denote by K_k a complete graph on k vertices. Any graph homomorphism $\Gamma \to K_k$ defines a proper k-colouring of the vertices of Γ; so if there exists a graph homomorphism $\Gamma \to K_k$, then the minimum such k is the chromatic number $\chi(\Gamma)$.

It is clear that the composite of two graph homomorphisms (when defined) is a graph homomorphism. This has the following easy consequence.

4.1 Lemma. *Suppose $\phi : \Gamma \to \Gamma'$ is a graph homomorphism. Then $\chi(\Gamma) \leqslant \chi(\Gamma')$.*

Proof. We may assume that $k = \chi(\Gamma') < \infty$ and that $\psi : \Gamma' \to K_k$ is a graph homomorphism. The composite map $\Gamma \xrightarrow{\phi} \Gamma' \xrightarrow{\psi} K_k$ is a graph homomorphism, so $\chi(\Gamma) \leqslant k$. $\square$

4.2 Lemma. *Let A be an additive group, on whose elements is defined a translation-invariant graph (i.e. for all $x, y, z \in A$, if x is adjacent to y then $x+z$ is adjacent to $y+z$). Then*

(i) $\chi(A) = \chi(A_0)$ where A_0 is the connected component of A containing the identity $0 \in A$.

(ii) Let B be another such additive group on which a translation-invariant graph is defined. Suppose $\phi : A \to B$ is additive (i.e. a group homomorphism) and that ϕ maps neighbours of $0 \in A$ to neighbours of $0 \in B$. Then $\chi(A) \leqslant \chi(B)$.

Proof. Any two connected components of A are related by a translation of A and hence are isomorphic. Moreover, each connected component may be coloured independently of the others. This proves (i), and (ii) follows from Lemma 4.1. $\square$

5. Taming the Transcendentals

Our main goal in this section is to prove Theorem 1.2(i). First, as a warmup exercise, we show the following related fact, whose proof is slightly less technical: $\chi(L^2) = \chi(K^2)$ for every purely transcendental extension $L \supseteq K$ of subfields of $\mathbb{R}$. This follows by induction from the following result (in fact by a finite induction in the case L is finitely generated over K; and this is the case we really care about, in view of the de Bruijn-Erdős result).

5.1 Lemma. *Let $L \supseteq K$ be subfields of $\mathbb{R}$, and suppose $L = K(\eta)$ where $\eta \in L$ is transcendental over K. Then $\chi(L^2) = \chi(K^2)$.*

Proof. Let $(\alpha_1, \alpha_2) \in L^2$ be a neighbour of $(0, 0)$, i.e. $\alpha_1^2 + \alpha_2^2 = 1$. We may express $\alpha_j = f_j(\eta)/g(\eta)$ for some polynomials $f_1(X), f_2(X), g(X) \in K[X]$ such that $f_1(X)^2 + f_2(X)^2 = g(X)^2$. Moreover we may assume $f_1(X), f_2(X), g(X)$ are pairwise relatively prime. Now if X divides $g(X)$, then since $K \subseteq \mathbb{R}$, $f_1(0)^2 + f_2(0)^2 = g(0)^2 = 0$ implies that $f_1(0) = f_2(0)$ so that X divides $f_1(X)$ and $f_2(X)$ also. This is impossible, so X does not divide $g(X)$.

By induction, it follows that every point of L^2 in L_0^2, the connected component of $(0, 0)$, is of the form $(f_1(\eta)/g(\eta),\ f_2(\eta)/g(\eta))$ for some $f_1(X), f_2(X), g(X) \in K[X]$ such that $g(0) \neq 0$. We therefore have a well-defined map $\phi : L_0^2 \to K^2$ given by

$$(f_1(\eta)/g(\eta),\ f_2(\eta)/g(\eta)) \mapsto (f_1(0)/g(0),\ f_2(0)/g(0)).$$

It is easy to see that ϕ maps adjacent vertices of L_0^2 to adjacent vertices of K^2. By Lemmas 4.1 and 4.2, we obtain $\chi(L^2) = \chi(L_0^2) \leqslant \chi(K^2)$. Since $\chi(K^2) \leqslant \chi(L^2)$ also, the result follows. $\square$

Finally, we prove that $\chi(\mathbb{R}^2) = \chi(K^2)$ for some number field $K \subset \mathbb{R}$, as claimed in Theorem 1.2(i). Recall that there exists a subfield $L \subset \mathbb{R}$ which is finitely generated over $\mathbb{Q}$, such $\chi(L^2) = \chi(\mathbb{R}^2)$. By the Noether Normalization Lemma, there exists a subfield

$E \subseteq L$ such that $[L : E] < \infty$, and the extension $L \supseteq \mathbb{Q}$ is finitely generated and purely transcendental. Let $e \geqslant 1$ be the transcendence degree of $E \supseteq \mathbb{Q}$, and let $\eta_1, \eta_2, \ldots, \eta_e \in E$ be a minimal set of generators for E over $\mathbb{Q}$. The ring $\mathcal{O} = \mathbb{Q}[\eta_1, \eta_2, \ldots, \eta_e]$ is isomorphic (as a $\mathbb{Q}$-algebra) to the polynomial ring $\mathbb{Q}[X_1, X_2, \ldots, X_e]$ under the evaluation map $X_j \mapsto \eta_j$. Moreover, E is the quotient field of $\mathcal{O}$.

There exists $\lambda \in L$ such that $L = E[\lambda]$ (see [8,p.34]). Let $h(Y) \in E[Y]$ be the minimal (monic) polynomial such that $h(\lambda) = 0$, i.e. the unique irreducible monic polynomial in $E[Y]$ having λ as a zero. Choose a nonzero polynomial $d(X_1, X_2, \ldots, X_e) \in \mathcal{O}[X_1, X_2, \ldots, X_e]$ such that

$$0 \neq d(\eta_1, \eta_2, \ldots, \eta_e) \in \mathcal{O} \quad \text{and} \quad d(\eta_1, \eta_2, \ldots, \eta_e)h(Y) \in \mathcal{O}[Y];$$

this simply means that $d(\eta_1, \ldots, \eta_e)$ is a common denominator for all coefficients in the polynomial $h(Y) \in E[Y]$. We therefore have

$$d(\eta_1, \eta_2, \ldots, \eta_e)h(Y) = \theta(\eta_1, \ldots, \eta_e, Y)$$

for some nonzero polynomial $\theta(X_1, X_2, \ldots, X_e, Y) \in \mathbb{Q}[X_1, X_2, \ldots, X_e, Y]$.

Let $\Gamma \subset K^2$ be a finite subgraph such that $\chi(\Gamma) = \chi(K^2) = \chi(\mathbb{R}^2)$. The vertices $P_1, P_2, \ldots, P_k \in \Gamma$ have the form

$$P_j = \left(\frac{\alpha_j}{\gamma}, \frac{\beta_j}{\gamma}\right), \quad j = 1, 2, \ldots, n$$

where $\alpha_j, \beta_j \in \mathcal{O}[\lambda]$ and $0 \neq \gamma \in \mathcal{O}$. (Thus $\gamma \in \mathcal{O}$ is a common denominator for the coordinates of all P_j.) We have

$$\alpha_j = a_j(\eta_1, \ldots, \eta_e, \lambda), \; \beta_j = b_j(\eta_1, \ldots, \eta_e, \lambda), \; \gamma = c(\eta_1, \ldots, \eta_e)$$

for some

$$a_j(X_1, \ldots, X_e, Y), \; b_j(X_1, \ldots, X_e, Y) \in \mathbb{Q}[X_1, \ldots, X_e, Y];$$
$$c(X_1, \ldots, X_e) \in \mathbb{Q}[X_1, \ldots, X_e].$$

Choose $r_1, r_2, \ldots, r_e \in \mathbb{Q}$ such that $c(r_1, r_2, \ldots, r_e) \neq 0$ and the polynomial $\theta(r_1, r_2, \ldots, r_e, Y) \in \mathbb{Q}[Y]$ is irreducible over $\mathbb{Q}$. The existence of r-tuples $(r_1, r_2, \ldots, r_e) \in \mathbb{Q}^e$ satisfying both of these conditions is assured; in fact most rational r-tuples work. To see this, first note that the set of all $(z_1, z_2, \ldots, z_e) \in \mathbb{C}^e$ for which $c(z_1, z_2, \ldots, z_e) = 0$ is a Zariski-closed proper subset $V_1 \subset \mathbb{C}^e$. Similarly, by Hilbert's Irreducibility Theorem (see e.g. [9, Chapter 1]), the set of all $(z_1, z_2, \dots, z_e) \in \mathbb{C}^e$ such that the polynomial $\theta(z_1, z_2, \dots, z_e, Y) \in \mathbb{Q}[Y]$ is reducible over $\mathbb{Q}$, is a proper Zariski-closed subset $V_2 \subset \mathbb{C}^e$. Since $V_1 \cup V_2 \subset \mathbb{C}^e$ is a proper Zariski-closed subset, its complement contains (infinitely many) rational points. Choose one such $(r_1, r_2, \dots, r_e) \in \mathbb{Q}^e \setminus (V_1 \cup V_2)$.

Let $K = \mathbb{Q}[\kappa]$ where κ is a zero of $\theta(r_1, r_2, \dots, r_e, Y) \in \mathbb{Q}[Y]$. We have a $\mathbb{Q}$-algebra homomorphism defined by

$$\begin{aligned} \mathbb{Q}[\eta_1, \eta_2, \dots, \eta_e, \lambda] = \mathcal{O}[\lambda] &\to \mathbb{Q}[\kappa] = K \\ f(\eta_1, \eta_2, \dots, \eta_e, \lambda) &\mapsto f(r_1, r_2, \dots, r_e, \kappa). \end{aligned}$$

It is easy to check that this induces a graph homomorphism from Γ into L^2. Indeed, if $P_j \sim P_k$ in Γ, then

$$(\alpha_j - \alpha_k)^2 + (\beta_j - \beta_k)^2 = \gamma^2,$$

i.e.

$$\begin{aligned} \big(a_j(\eta_1, \dots, \eta_e, \lambda) - a_k(\eta_1, \dots, \eta_e, \lambda)\big)^2 + \big(b_j(\eta_1, \dots, \eta_e, \lambda) - b_k(\eta_1, \dots, \eta_e, \lambda)\big)^2 \\ = c(\eta_1, \dots, \eta_e)^2. \end{aligned}$$

This means that

$$\begin{aligned} \big(a_j(\eta_1, \dots, \eta_e, Y) - a_k(\eta_1, \dots, \eta_e, Y)\big)^2 + \big(b_j(\eta_1, \dots, \eta_e, Y) - b_k(\eta_1, \dots, \eta_e, Y)\big)^2 \\ - c(\eta_1, \dots, \eta_e)^2 \;=\; q(Y)\theta(\eta_1, \dots, \eta_e, Y) \end{aligned}$$

for some $q(Y) \in \mathbb{Q}[Y]$. This in turn implies that

$$\begin{aligned} \big(a_j(X_1, \dots, X_e, Y) - a_k(X_1, \dots, X_e, Y)\big)^2 + \big(b_j(X_1, \dots, X_e, Y) - b_k(X_1, \dots, X_e, Y)\big)^2 \\ - c(X_1, \dots, X_e)^2 \;=\; q(Y)\theta(X_1, \dots, X_e, Y) \end{aligned}$$

and so

$$\begin{aligned} \big(a_j(r_1, \dots, r_e, Y) - a_k(r_1, \dots, r_e, Y)\big)^2 + \big(b_j(r_1, \dots, r_e, Y) - b_k(r_1, \dots, r_e, Y)\big)^2 \\ - c(r_1, \dots, r_e)^2 \;=\; q(Y)\theta(r_1, \dots, r_e, Y) \end{aligned}$$

whence

$$\begin{aligned} \big(a_j(r_1, \dots, r_e, \kappa) - a_k(r_1, \dots, r_e, \lambda)\big)^2 + \big(b_j(r_1, \dots, r_e, \kappa) - b_k(r_1, \dots, r_e, \kappa)\big)^2 \\ = c(r_1, \dots, r_e)^2, \end{aligned}$$

i.e.

$$\Big(\frac{a_j(r_1,\ldots,r_e,\kappa)}{c(r_1,\ldots,r_e)},\ \frac{a_k(r_1,\ldots,r_e,\kappa)}{c(r_1,\ldots,r_e)}\Big) \sim \Big(\frac{a_j(r_1,\ldots,r_e,\kappa)}{c(r_1,\ldots,r_e)},\ \frac{a_k(r_1,\ldots,r_e,\kappa)}{c(r_1,\ldots,r_e)}\Big)$$

in L^2. By Lemma 4.1,

$$\chi(\mathbb{R}^2) = \chi(L^2) = \chi(\Gamma) \leqslant \chi(K^2) \leqslant \chi(\mathbb{R}^2)$$

and so equality must hold throughout. □

6. Finite Fields

Let q be a prime power. The graph $\mathbb{F}_q^2$, with adjacency relation defined by (1.1) as before, has q^2 vertices, and is regular of degree

$$\begin{cases} q, & \text{for } q \text{ even;} \\ q-1, & \text{if } q \equiv 1 \mod 4; \\ q+1, & \text{if } q \equiv 3 \mod 4; \end{cases}$$

see [5,p.93]. With some computer assistance we obtain some information on the chromatic number for small values of q:

6.1 Table. Chromatic number of $\mathbb{F}_q^2$ for small q

q	2	3	4	5	7	8	9	11	13	16	17
$\chi(\mathbb{F}_q^2)$	2	3	2	3	4	2	3	5	5 or 6	2	5, 6 or 7

For example, in the cases $K = \mathbb{F}_3$ or $\mathbb{F}_7$, optimal proper colourings are given by the respective arrays

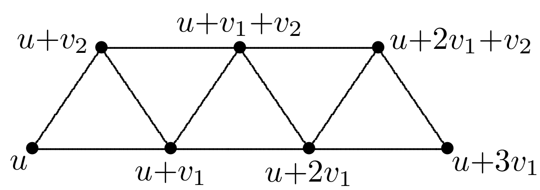
$$\begin{matrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{matrix} \qquad \text{and} \qquad \begin{matrix} 0 & 1 & 3 & 1 & 2 & 0 & 2 \\ 1 & 2 & 0 & 2 & 0 & 1 & 3 \\ 2 & 0 & 1 & 3 & 1 & 2 & 0 \\ 3 & 1 & 2 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 3 & 1 & 2 \\ 1 & 3 & 1 & 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 & 1 & 3 & 1 \end{matrix} \ .$$

6.2 Lemma. *(a) If q is even, then $\mathbb{F}_q^2$ is a union of $q/2$ disjoint complete bipartite graphs $K_{q,q}$, and $\chi(\mathbb{F}_q^2) = 2$.*

(b) *If q is odd, then $\chi(\mathbb{F}_q^2) \geqslant 3$. If $q \equiv \pm 1 \mod 12$, then $\chi(\mathbb{F}_q^2) \geqslant 4$.*

Proof. (a) See Proposition 3.2. Indeed for q be even, there exists an $\mathbb{F}_2$-linear map $\theta : \mathbb{F}_q \to \mathbb{F}_2$ such that $\theta(1) = 1$; one checks that the map $\mathbb{F}_q^2 \to \mathbb{F}_2$ given by $(x, y) \mapsto \theta(x+y)$ is a proper 2-colouring.

(b) Suppose $q = p^e$ is odd. Then the p points $(a, 0) \in \mathbb{F}_q^2$ for $a \in \mathbb{F}_p$ form a cycle of odd length p. Now suppose that $q \equiv \pm 1 \mod 12$, so that $\mathbb{F}_q$ contains $\sqrt{3}$. Let $v_1 = (1, 0) \in \mathbb{F}_q^2$, $v_2 = \frac{1}{2}(1, \sqrt{3}) \in \mathbb{F}_q^2$ and let $u \in \mathbb{F}_q^2$ be arbitrary. Consider the subgraph of $\mathbb{F}_q^2$ shown. In any proper 3-colouring of $\mathbb{F}_q^2$, it is easy to see that the vertices u and $u + 3v_1$ must have the same colour.



By induction, it follows that vertices u and $u + 3kv_1$ have the same colour for each integer k; but then choosing $k \in \mathbb{Z}$ such that $3k \equiv 1 \mod p$ gives a contradiction. $\square$

It seems likely that for odd prime powers q, the value of $\chi(\mathbb{F}_q^2)$ should tend to ∞ as $q \to \infty$; this guess is supported by Table 6.1.

7. Extensions of Odd Degree

The following fact was cited twice in Section 1.

7.1 Theorem. *Let $K \supseteq \mathbb{Q}$ be a finite extension of odd degree. Then $\chi(K^2) = 2$. (In particular, $\chi(\mathbb{Q}^2) = 2$.)*

Proof. Clearly $\chi(K^2) \geqslant 2$. Let $\mathcal{O}$ be the ring of algebraic integers in K. Consider the prime factorization of the ideal $2\mathcal{O}$ in $\mathcal{O}$:

$$2\mathcal{O} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_k^{e_k}$$

where $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_k$ are the distinct prime ideals of $\mathcal{O}$ containing 2. Now $\mathcal{O}/\mathfrak{p}_j \cong \mathbb{F}_{2^{f_j}}$ for some integers $f_j \geqslant 0$. Since

$$e_1 f_1 + e_2 f_2 + \cdots + e_k f_k = [K : \mathbb{Q}]$$

is odd, at least one of the terms $e_j f_j$ is odd. Fix such an index j and write $\mathfrak{p} = \mathfrak{p}_j$, $e = e_j$, $f = f_j$. Let R be the subring of K consisting of all fractions a/b with $a, b \in \mathcal{O}$ and $b \notin \mathfrak{p}$. Then R is a local ring with unique maximal ideal $R\mathfrak{p}$ and residue field

$$R/R\mathfrak{p} \cong \mathcal{O}/\mathfrak{p} \cong E := \mathbb{F}_{2^f} .$$

In view of Lemma 4.2, the theorem will follow from the following two facts, which we will justify below:

(i) The connected component of $(0,0)$ in K^2 lies in R^2.

(ii) There is a proper colouring $\psi : R^2 \to F$ where $F = \mathbb{F}_2$.

Consider any neighbour of $(0,0)$ in K^2, which we may write as $(\alpha/\gamma, \beta/\gamma)$ where $\alpha, \beta, \gamma \in \mathcal{O}$. Suppose that $\gamma \in \mathfrak{p}$. By Lemma 2.3, we may assume that neither α nor β lies in $\mathfrak{p}$. Since

$$\nu_{\mathfrak{p}}(\alpha+\beta+\gamma) + \nu_{\mathfrak{p}}(\alpha+\beta-\gamma) = \nu_{\mathfrak{p}}((\alpha+\beta+\gamma)(\alpha+\beta-\gamma)) = \nu_{\mathfrak{p}}(2\alpha\beta) = \nu_{\mathfrak{p}}(2) = e$$

is odd, we have

$$\nu_{\mathfrak{p}}(\alpha+\beta+\gamma) \neq \nu_{\mathfrak{p}}(\alpha+\beta-\gamma) \quad \text{and} \quad min\{\nu_{\mathfrak{p}}(\alpha+\beta+\gamma), \nu_{\mathfrak{p}}(\alpha+\beta-\gamma)\} \leqslant \frac{e-1}{2} .$$

Thus

$$\begin{aligned} e+1 &\leqslant e + \nu_{\mathfrak{p}}(\gamma) = \nu_{\mathfrak{p}}(2\gamma) \\ &= \nu_{\mathfrak{p}}((\alpha+\beta+\gamma) - (\alpha+\beta-\gamma)) \\ &= min\{\nu_{\mathfrak{p}}(\alpha+\beta+\gamma), \nu_{\mathfrak{p}}(\alpha+\beta-\gamma)\} \\ &\leqslant (e-1)/2, \end{aligned}$$

a contradiction. This verifies our claim that $\gamma \notin \mathfrak{p}$, so every neighbour of $(0,0)$ lies in R^2. Now (i) follows by induction.

Let $Tr : E \to F = \mathbb{F}_2$ denote the trace map. Since $[E : F] = f$ is odd, we have $Tr(1) = 1$. Also $Tr(x^2) = Tr(x)$ for all $x \in E$ since $x \mapsto x^2$ is an automorphism of E. Now define $\psi : R^2 \to F$ by $\psi(\xi, \eta) = Tr(\xi + \eta + R\mathfrak{p})$ where $\xi + \eta + R\mathfrak{p} \in R/R\mathfrak{p} = E$. If $(\xi, \eta) \in R^2$ is a neighbour of $(0,0)$ then $\xi^2 + \eta^2 = 1$; writing $x, y \in E$ for the reductions of $\xi, \eta \in R$ modulo $R\mathfrak{p}$ respectively, then

$$\psi(\xi, \eta) = Tr(x + y) = Tr(x^2 + y^2) = Tr(1) = 1.$$

Since $\psi : R^2 \to F$ is additive, (ii) follows by induction. $\square$

The verification of (ii) in the latter proof, may be viewed as another application of Lemma 4.1. Namely, if we consider F as a graph on two vertices with a single edge, then $\phi : R^2 \to F$ is a graph homomorphism. This point of view provides the key to the next section.

8. Quadratic Extensions of $\mathbb{Q}$

Consider a real quadratic extension of $\mathbb{Q}$, i.e. a field of the form $K = \mathbb{Q}(\sqrt{d})$ for some squarefree integer $d \geqslant 2$. By combining Corollary 8.3 and Lemma 8.4 below, we obtain the following upper bound for $\chi(K^2)$. For a lower bound, see Theorem 8.6.

8.1 Theorem. *If $d \not\equiv 47$, 59 or 83 mod 84, then $\chi(K^2) \leqslant 4$.*

The ring of algebraic integers in K is

$$\mathcal{O} = \begin{cases} \mathbb{Z}[\sqrt{d}], & d \equiv 2 \text{ or } 3 \mod 4; \\ \mathbb{Z}[(1+\sqrt{d})/2], & d \equiv 1 \mod 4. \end{cases}$$

Each prime $p \in \mathbb{Z}$ either ramifies, splits, or remains prime in $\mathcal{O}$, depending on the choice of d and p; see [8, p.75].

8.2 Lemma. *Suppose we have a prime $p \equiv 3 \mod 4$ for which $d \equiv 0 \mod p$ or d is a quadratic residue modulo p. Then $\chi(K^2) \leqslant \chi(\mathbb{F}_p^2)$.*

Proof. By hypothesis, p either ramifies or splits in $\mathcal{O}$; that is, $p\mathcal{O} = \mathfrak{p}\mathfrak{p}'$ for some (not necessarily distinct) ideals $\mathfrak{p}, \mathfrak{p}' \subset \mathcal{O}$ of norm p. Let $R \subset \mathcal{O}$ be the subring consisting of all fractions a/b such that $a, b \in \mathcal{O}$ and $b \notin \mathfrak{p}$. Then $R/R\mathfrak{p} \cong \mathcal{O}/\mathfrak{p} \cong \mathbb{F}_p$.

Consider any neighbour of $(0,0)$ in K^2, which is necessarily of the form $(\alpha/\gamma, \beta/\gamma)$ for some $\alpha, \beta, \gamma \in \mathcal{O}$ such that $\alpha^2 + \beta^2 = \gamma^2$. By Lemma 2.3, we may assume that no two of α, β, γ belong to $\mathfrak{p}$. If $\gamma \in \mathfrak{p}$ then $\alpha^2 + \beta^2 \equiv 0 \mod \mathfrak{p}$, so that -1 is a square in $\mathcal{O}/\mathfrak{p} \cong \mathbb{F}_p$. This is impossible since $p \equiv 3 \mod 4$. Therefore $\gamma \notin \mathfrak{p}$ and the reduction of $(\alpha/\gamma,\ \beta/\gamma)$ modulo $\mathfrak{p}$ gives neighbour of $(0,0)$ in $\mathbb{F}_p^2$. We see now that K_0^2, the connected component of K^2 containing $(0,0)$, is contained in R^2, and that the reduction modulo $\mathfrak{p}$ induces a graph homomorphism $K_0^2 \to \mathbb{F}_p^2$. The result follows by Lemmas 4.1 and 4.2. $\square$

Since $\chi(\mathbb{F}_3^2) = 3$ and $\chi(\mathbb{F}_7^2) = 4$ (see Section 7), this yields:

8.3 Corollary. *If* $d \equiv 0$ *or* $1 \mod 3$, *then* $\chi(K^2) \leqslant 3$. *If* $d \equiv 0$, *1, 2 or* $4 \mod 7$, *then* $\chi(K^2) \leqslant 4$. $\square$

A similar, but more delicate argument, works with $p = 2$:

8.4 Lemma. *If* $d \equiv 1 \mod 4$, *then* $\chi\big(\mathbb{Q}(\sqrt{d})^2\big) = 2$.

Proof. First suppose that $d \equiv 1 \mod 8$, so that 2 splits: $2\mathcal{O} = \mathfrak{p}\mathfrak{p}'$ for prime ideals $\mathfrak{p} \neq \mathfrak{p}'$. Now $\mathcal{O}/\mathfrak{p} \cong \mathbb{F}_2$ and $\mathcal{O}/\mathfrak{p}^2 \cong \mathbb{Z}/4\mathbb{Z}$. Consider a neighbour $(\alpha/\gamma,\, \beta/\gamma)$ of $(0,0)$ in K^2, where $\alpha, \beta, \gamma \in \mathcal{O}$ and $\alpha^2 + \beta^2 = \gamma^2$. If $\gamma \in \mathfrak{p}$ then we may suppose $\alpha, \beta \notin \mathfrak{p}$, so that $\alpha, \beta \equiv \pm 1 \mod \mathfrak{p}$ and $\alpha^2 + \beta^2 \equiv 2 \not\equiv 0 \equiv \gamma^2 \mod \mathfrak{p}$, a contradiction. Thus $\gamma \notin \mathfrak{p}$ and $(\alpha/\gamma,\, \beta/\gamma) \in R^2$ where $R = \mathcal{O}_{\mathfrak{p}}$. Reduction modulo $\mathfrak{p}$ gives a homomorphism $R^2 \to \mathbb{F}_2^2$ so that $\chi(K^2) \leqslant \chi(R^2) \leqslant \chi(\mathbb{F}_2^2) = 2$.

In the other case $d \equiv 5 \mod 8$, and 2 remains prime: $\mathcal{O}/2\mathcal{O} \cong \mathbb{F}_4$. Consider a neighbour $(\alpha/\gamma,\, \beta/\gamma)$ of $(0,0)$ in K^2, where $\alpha, \beta, \gamma \in \mathcal{O}$ and $\alpha^2 + \beta^2 = \gamma^2$. If $\gamma \in 2\mathcal{O}$ then we may suppose $\alpha, \beta \notin 2\mathcal{O}$ and $\alpha^2 + \beta^2 \equiv 2 \not\equiv 0 \equiv \gamma^2 \mod 4\mathcal{O}$, a contradiction. Thus $\gamma \notin 2\mathcal{O}$ and $(\alpha/\gamma,\, \beta/\gamma) \in R^2$ where $R = \mathcal{O}_{2\mathcal{O}}$. Reduction modulo 2 gives a homomorphism $R^2 \to \mathbb{F}_4^2$ so that $\chi(K^2) \leqslant \chi(R^2) \leqslant \chi(\mathbb{F}_4^2) = 2$. $\square$

The following is obtained by considering a homomorphic image of the form R^2 where R is no longer a field, but rather a commutative ring of order 8 with unity.

8.5 Theorem. *If* $d \equiv 2 \mod 4$ *then* $\chi(K^2) = 2$.

Proof. In this case the ring of algebraic integers in K is given by $\mathcal{O} = \mathbb{Z}[\theta]$ where $\theta = \sqrt{d}$, and the rational prime 2 ramifies: $2\mathcal{O} = \mathfrak{p}^2$ where $\mathfrak{p} = 2\mathcal{O} + \theta\mathcal{O}$. Consider a neighbour of $(0,0)$ in K^2, which by Lemma 2.3 is expressible as $(a/c, b/c)$ where $a, b, c \in \mathcal{O}$, $a^2 + b^2 = c^2 \neq 0$, and at most one of a, b, c lies in $\mathfrak{p}$.

We claim that $\nu_{\mathfrak{p}}(c) \leqslant 1$. For suppose that $c \in \mathfrak{p}$, and so necessarily $\nu_{\mathfrak{p}}(a) = \nu_{\mathfrak{p}}(b) = 0$. Now $a \equiv \pm 1 + a_1\theta \mod 4\mathcal{O}$ for some $a_1 \in \{0, 1, 2, 3\}$, and examination of cases reveals $a^2 \equiv 1$ or $3{+}2\theta \mod 4\mathcal{O}$, so that $a^2{+}b^2 \equiv 2$ or $2\theta \mod 4\mathcal{O}$; in particular $2\nu_{\mathfrak{p}}(c) = \nu_{\mathfrak{p}}(c^2) = \nu_{\mathfrak{p}}\big(a^2{+}b^2\big) \leqslant 3$ and $\nu_{\mathfrak{p}}(c) \leqslant 1$ as claimed. Thus all vertices of K_0^2, the connected component of $(0,0)$ in K^2, has vertices in the additive subgroup $S = \{x \in K : \nu_{\mathfrak{p}}(x) \geqslant -1\} \subset K$.

Let $A = S/\mathfrak{p}^3S$, an additive group of order 8 consisting of the cosets of $\mathfrak{p}^3S$ represented by 0, θ^{-1}, $2\theta^{-1}$, $3\theta^{-1}$, 1, $1{+}\theta^{-1}$, $1{+}2\theta^{-1}$, $1{+}3\theta^{-1}$. Consider the graph on $A^2 = A \times A$ with adjacency relation defined as usual: two vertices $(\alpha,\beta),(\alpha',\beta') \in A^2$ are adjacent iff $(\alpha'{-}\alpha)^2 + (\beta'{-}\beta)^2 = 1$. Note that the latter equality is well-defined in A (as congruence mod $\mathfrak{p}^3S$). Now it is easy to see that the natural homomorphism $S \to A$ induces a graph homomorphism $K^2_{(0,0)} \to A^2$, and so $\chi(K^2) \leqslant \chi(A^2)$ by Lemma 4.1.

It is straightforward to check, however, that A^2 is bipartite. Indeed, if we let $B \subset A^2$ be the additive subgroup of order 8 generated by the elements $(1, 1{+}\theta)$, $(0,2)$ and $(2,0)$, then B has four additive cosets

$$B, \quad (0,\theta){+}B, \quad (0,1){+}B \quad \text{and} \quad (0,1{+}\theta){+}B$$

in A^2, and every edge in A^2 extends either between B and $(0,\theta){+}B$, or between $(0,1){+}B$ and $(0,1{+}\theta){+}B$. The result follows. $\square$

8.6 Theorem. *If K contains $\sqrt{p}$ for some prime $p \equiv 3 \mod 4$, then $\chi(K^2) \geqslant 3$.*

Proof. Let (a,b) be a minimal positive solution of Pell's equation $a^2 - pb^2 = 1$; that is, $a > b > 0$ with a (and b) as small as possible. It is known that a is even. To see this, note that if a is odd then $gcd(a+1, a-1) = 2$ and $(a+1)(a-1) = pb^2$ implies that $\{a+1, a-1\} = \{2u^2, 2pv^2\}$ for some integers u, v with $u^2 - pv^2 = 1$, contradicting the minimality of the solution (a,b).

Now the isosceles triangle with vertices $(0,0)$ and $\frac{1}{2}\big(b\sqrt{p}, \pm 1\big)$ has integer sides $\frac{a}{2}$, $\frac{a}{2}$ and 1. Its perimeter constitutes a cycle of odd length $a+1$ in K^2. $\square$

9. Relationship with Hedetniemi's Conjecture

Given two graphs Γ and Γ', the *direct product* $\Gamma \times \Gamma'$ is the graph whose vertex set is the Cartesian product of the vertex sets of Γ and of Γ'; and with adjacency defined by $(x,x') \sim (y,y')$ iff $x \sim y$ and $x' \sim y'$. In other words, an adjacency matrix for $\Gamma \times \Gamma'$ is given by $A \otimes A'$ where A, A' are adjacency matrices for Γ, Γ' respectively. Clearly in general, (9.1) $\chi(\Gamma \times \Gamma') \leqslant min\{\chi(\Gamma), \chi(\Gamma')\}$.

To see this, note that projection onto the first coordinate gives a graph homomorphism $\Gamma \times \Gamma' \to \Gamma$, so $\chi(\Gamma \times \Gamma') \leqslant \chi(\Gamma)$ by Lemma 4.1; and $\chi(\Gamma \times \Gamma') \leqslant \chi(\Gamma')$ similarly. This suggests the question

(9.2) Must equality hold in (9.1) for all graphs Γ, Γ'?

Hedetniemi's Conjecture says that the answer to (9.2) is affirmative. See [6,pp.180–181] for a survey of progress on this open problem. In particular, (9.2) has been answered affirmatively in the special case $\chi(\Gamma), \chi(\Gamma') \leqslant 4$. Question (9.2) is related to a natural generalization of the technique of Sections 7–8, as we proceed to describe.

Once again consider a finite real extension $K \supseteq \mathbb{Q}$, with ring of algebraic integers $\mathcal{O}$, and let $\mathfrak{B} \subset \mathcal{O}$ be an arbitrary ideal with prime factorization given by

$$\mathfrak{B} = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2}\cdots\mathfrak{p}_k^{e_k}.$$

Set

$$R := \mathcal{O}/\mathfrak{B} \cong \left(\mathcal{O}/\mathfrak{p}_1^{e_1}\right) \oplus \left(\mathcal{O}/\mathfrak{p}_2^{e_2}\right) \oplus \cdots \oplus \left(\mathcal{O}/\mathfrak{p}_k^{e_k}\right).$$

The graph formed on the vertex set $R^2 = R \times R$ by the relation (1.1) is the direct product of the graphs $\mathcal{O}/\mathfrak{p}_j^{e_j}$ for $j = 1, 2, 3, \ldots, k$. In the spirit of Sections 7 and 8, it is natural to choose our ideal $\mathfrak{B}$ such that all neighbours of $(0,0)$ in K^2 have the form $(\alpha/\gamma, \beta/\gamma)$ where $\alpha^2 + \beta^2 = \gamma^2$ such that $\gamma \notin \mathfrak{B}$; for then

$$\chi(K^2) \leqslant \chi((\mathcal{O}/\mathfrak{B})^2) \leqslant \min\{\chi((\mathcal{O}/\mathfrak{p}_j^{e_j})^2) : j = 1, 2, 3, \ldots, k\}.$$

This would at first seem to offer an improvement over the results of Sections 7 and 8 where all such ideals $\mathfrak{B}$ considered were either prime or the square of a prime ideal. However, to obtain such an improvement would require producing a counterexample to Hedetniemi's Conjecture.

10. Colouring $\mathbb{R}^n$ and $\mathbb{C}^n$

Consider more generally the graph defined on K^n with adjacency relation

(10.1) $(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n)$ iff $(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2 = 1$.

It is not hard to see that $\chi(\mathbb{R}^n) < \infty$, generalizing the proof that $\chi(\mathbb{R}^2) \leqslant 7$; see also [3] for $\chi(\mathbb{R}^3) \leqslant 18$. To obtain a finite upper bound for $\chi(\mathbb{R}^n)$, first choose any lattice $L \subset \mathbb{R}^n$ (preferably a lattice corresponding to a dense sphere-packing). For each $x \in L$, let V_x be the Voronoi cell of L with center x (see [2,p.33]). Each such cell has the same diameter, say δ. It is not hard to show that for some sublattice $L_1 \subset L$, we have $d(V_x, V_y) \geqslant \varepsilon > \delta$ for all $x \neq y$ in L_1; here d denotes Euclidean distance. We may assume (after scaling as necessary) that $\varepsilon > 1 > \delta$. Choose one colour for each coset $L_1 + u$, for $u \in L$, and colour all points of $\bigcup_{x \in L_1+u} V_x$ with this colour. Then $\chi(\mathbb{R}^n) \leqslant [L : L_1] < \infty$.

The case for subfields $K \subseteq \mathbb{C}$ is much different. I do not know of any finite upper bound for $\chi(K^n)$, or even $\chi(K^2)$, in this case. Moreover our proof of Lemma 5.1 strongly used the fact that we have subfields of $\mathbb{R}$. In the case of complex fields, one's first thought might be to replace (10.1) by the relation

$$(10.2) \qquad (x_1, \ldots, x_n) \approx (y_1, \ldots, y_n) \text{ iff } |x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots + |x_n - y_n|^2 = 1.$$

However, on second thought we see that this alternative adjacency relation is less interesting, since it ignores the complex structure of $\mathbb{C}^n$, failing to distinguish it from $\mathbb{R}^{2n}$.

Observe that the adjacency relation (10.1) on $\mathbb{C}^n$ yields 'fewer' edges than does (10.2), in the sense of topological dimension: With (10.1), the neighbours of each point form a real $(2n-2)$-manifold, albeit unbounded; whereas with (10.2), the neighbours of each point form a real $(2n-1)$-manifold S^{2n-1}. From this point of view, it is perhaps surprising that while (10.2) yields a graph with clearly finite chromatic number, whereas with (10.1), this is unclear.

Felix Lazebnik recently asked me: for which subfields $K \subseteq \mathbb{C}$ can I say for sure that $\chi(K^2) < \infty$? Of course this holds if K has at least one embedding in $\mathbb{R}$. In the case of algebraic number fields, we obtain the following stronger result.

10.3 Theorem. *Let $K \supseteq \mathbb{Q}$ be a finite extension. If K does not contain $i = \sqrt{-1}$, then $\chi(K^2) < \infty$.*

Proof. Let $\mathcal{O}$ be the ring of algebraic integers in K, and suppose that $i \notin K$. We first observe that

$$(10.4) \qquad \text{there exists a prime ideal } \mathfrak{p} \subset \mathcal{O} \text{ satisfying } |\mathcal{O}/\mathfrak{p}| \equiv 3 \mod 4.$$

For suppose that (10.4) fails. Then for every prime ideal $\mathfrak{p} \subset \mathcal{O}$, the polynomial $X^2 + 1$ factors in $\mathcal{O}/\mathfrak{p}$. By Hensel's Lemma (see e.g. [7,p.129]), it also factors in the completion

$K_{\mathfrak{p}}$. By the Hasse-Minkowski Theorem (see [7,p.385]), $X^2 + 1$ also factors in K, contrary to hypothesis.

Consider an ideal $\mathfrak{p} \subset \mathcal{O}$ as in (10.4), and let $q = |\mathcal{O}/\mathfrak{p}|$. We have seen that there exists a graph homomorphism $K^2 \to \mathbb{F}_q^2$ so that $\chi(K^2) \leqslant \chi(\mathbb{F}_q^2) < \infty$. $\square$

For subfields of $\mathbb{C}$ containing i, I am stymied. Put bluntly, I do not even know whether or not $\chi(\mathbb{Q}[i]^2)$ is finite.

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