Europ. J. Combinatorics **18** (1997), 171–173 8 January, 1996

The Non-existence of Ovoids in $O_9(q)$

Athula Gunawardena
Division of Math and Sciences, Wayne State College, Wayne NE 68787

and

G. Eric Moorhouse
Dept. of Mathematics, University of Wyoming, Laramie WY 82071

Abstract. We prove the nonexistence of ovoids in finite orthogonal spaces of type $O_{2n+1}(q)$ for $n \geq 4$.

Keywords: ovoid, two-graph

1. The Result

Let $V = F^m$, $F = GF(q)$, and let $Q : V \to F$ be a quadratic form. Thus $Q(\lambda x) = \lambda^2 Q(x)$, $Q(x+y) = Q(x)+Q(y)+f(x,y)$ for all $\lambda \in F$; $x, y \in V$, where f is bilinear. A *singular point* is a one-dimensional subspace $\langle v \rangle$ such that $Q(v) = 0$. We assume Q is *nondegenerate,* i.e. $V^\perp$ is either 0, or (if q is even and m odd) a nonsingular point. (Here $\perp$ denotes orthogonal with respect to f.) The isometry type of (V, Q) is denoted $O_{2n+1}(q)$, $O^+_{2n}(q)$ or $O^-_{2n}(q)$ according as $m = 2n + 1$, or $m = 2n$ and Q is hyperbolic or elliptic (i.e. maximal totally singular subspaces have dimension $m/2$ or $(m-2)/2$).

An *ovoid* in (V, Q) is a collection $\mathcal{O}$ consisting of singular points, such that every maximal totally singular subspace contains exactly one point of $\mathcal{O}$. In this section we prove:

Theorem. *For $n \geq 4$, $O_{2n+1}(q)$ has no ovoids.*

This result is known for q even (see [T]). Henceforth we suppose that $\mathcal{O}$ is an ovoid in $O_{2n+1}(q)$, q odd. Thus (see [HT], [K]) $\mathcal{O}$ consists of $q^n + 1$ singular points, no two of which are perpendicular with respect to f. For convenience, we may assume henceforth that $(-1)^n 2\delta$ is a nonsquare in F, where δ is the discriminant of Q; otherwise, replace Q by εQ where $\varepsilon \in F$ is a nonsquare.

Let Δ be the collection of all triples $\{\langle u \rangle, \langle v \rangle, \langle w \rangle\}$ of points of $\mathcal{O}$ such that $f(u, v)f(v, w)f(w, u)$ is a nonsquare of F. As shown in [M], every 4-subset of $\mathcal{O}$ contains an even number of triples from Δ; thus Δ is a *two-graph* with point set $\mathcal{O}$. We next show that any 2-subset of $\mathcal{O}$ is contained in exactly $\frac{1}{2}(q^{n-1} + 1)(q - 1)$ triples from Δ.

Lemma. *The two-graph Δ is regular of degree $\frac{1}{2}(q^{n-1}+1)(q-1)$.*

Proof. Suppose $\langle u\rangle$, $\langle v\rangle$, $\langle w\rangle$ are distinct points of $\mathcal{O}$. Then $\langle u,v,w\rangle$ is a plane on which Q restricts to a nondegenerate quadratic form with discriminant $-2f(u,v)f(v,w)f(w,u)$; thus $\langle u,v,w\rangle^\perp$ has discriminant $(-1)^{n-1}f(u,v)f(v,w)f(w,u)\eta$ for some nonsquare $\eta\in F$. In particular, $\langle u,v,w\rangle^\perp$ is isometric to $O^+_{2n-2}(q)$ or $O^-_{2n-2}(q)$, according as $f(u,v)f(v,w)f(w,u)$ is a nonsquare or a square in F.

Now fix $\langle u\rangle\neq\langle v\rangle$ in $\mathcal{O}$. Let N (resp., S) be the number of points $\langle w\rangle\in\mathcal{O}$ distinct from $\langle u\rangle$, $\langle v\rangle$ such that $f(u,v)f(v,w)f(w,u)$ is a nonsquare (resp., a square) in F. Clearly,

$$N+S=|\mathcal{O}|-2=q^n-1.$$

Counting in two different ways the number of pairs $(\langle w\rangle,\langle x\rangle)$ consisting of a point $\langle w\rangle\in\mathcal{O}$ distinct from $\langle u\rangle,\langle v\rangle$, and a singular point $\langle x\rangle$ in $\langle u,v,w\rangle^\perp$, we obtain

$$\frac{(q^{n-2}+1)(q^{n-1}-1)}{q-1}N+\frac{(q^{n-2}-1)(q^{n-1}+1)}{q-1}S=\frac{q^{2n-2}-1}{q-1}(q^{n-1}-1).$$

Here we have used the following facts (see [HT], p.23; [K]): $O^\pm_{2n-2}(q)$ has exactly $(q^{n-2}\pm 1)(q^{n-1}\mp 1)/(q-1)$ singular points; $\langle u,v\rangle^\perp\simeq O_{2n-1}(q)$ has exactly $(q^{2n-2}-1)/(q-1)$ singular points; and $x^\perp$ contains exactly $q^{n-1}+1$ points of $\mathcal{O}$. The unique solution is given by

$$N=\tfrac{1}{2}(q^{n-1}+1)(q-1),\qquad S=\tfrac{1}{2}(q^{n-1}-1)(q+1),$$

which verifies the Lemma. □

Observe that if Q is replaced by εQ, where $\varepsilon\in F$ is nonsquare, then $\mathcal{O}$ yields the complementary two-graph $\overline{\Delta}$ of degree $\frac{1}{2}(q^{n-1}-1)(q+1)$.

The eigenvalues $\rho_1>\rho_2$ of the $(0,\pm1)$-adjacency matrix of Δ (see [S], Theorem 7.2) satisfy

$$q^n+1=1-\rho_1\rho_2,\qquad \tfrac{1}{2}(q^{n-1}+1)(q-1)=-\tfrac{1}{2}(\rho_1+1)(\rho_2+1)$$

from which we obtain $\rho_1=q^{n-1}$, $\rho_2=-q$. The corresponding multiplicities μ_1,μ_2 satisfy $q^{n-1}\mu_1-q\mu_2=0$, $\mu_1+\mu_2=q^n+1$ (see [S], Theorem 7.7) and so

$$\mu_1=q^2-\frac{q^2-1}{q^{n-2}+1}$$

must be an integer. The proof of the Theorem follows.

2. Concluding Remarks

No ovoids are known in $O^+_{2n}(q)$ for $n \geq 5$. Some nonexistence results in this case are found in [BM].

There are just two families of known ovoids in $O_7(q)$: for each $q = 3^e$, an ovoid admitting $PGU(3,q)$ as a group of automorphisms; and for $q = 3^{2t+1}$, and ovoid admitting the Ree group ${}^2G_2(q)$. The resulting *unitary* and *Ree two-graphs* on q^3+1 points have degree $\frac{1}{2}(q^2+1)(q-1)$, in accordance with our Lemma. These are described in [S]. Note that the two families coincide for $q = 3$, resulting in the 2-transitive two-graph on 28 points admitting $Sp(6,2)$. It is known ([T], [OT]) that $O_7(q)$ posesses no ovoids for $q = 2^e, 5, 7$, but except for the values of q we mention here, the question of existence of ovoids in $O_7(q)$ remains open.

The $O^-_4(q)$ quadric embeds as an ovoid in $O_5(q)$, and for q odd, the resulting regular two-graph of degree $\frac{1}{2}(q^2-1)$ on q^2+1 vertices, is of Paley type (see [M]). However, more examples of ovoids in $O_5(q)$ are known [K]. By our Lemma, all such examples yield regular two-graphs with the same parameters.

In [M] the question was posed: can there exist nonisomorphic ovoids $\mathcal{O}, \mathcal{O}'$ in an orthogonal space of dimension $m \geq 6$, but with the same invariant two-graph? While we have not answered this question, the above examples show that it is possible for $\mathcal{O} \not\simeq \mathcal{O}'$ to give regular two-graphs with the same parameters, and hence the same fingerprint (cf. [M]). In all known occurrences of this phenomenon, q is nonprime.

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