Solutions to the December 2005 Putnam Exam

G. Eric Moorhouse, University of Wyoming
http://www.uwyo.edu/moorhouse/handouts/putnam05/

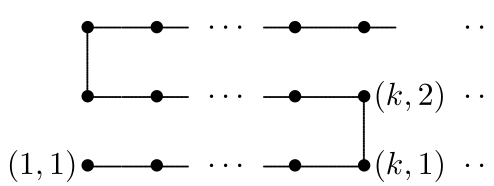
I'm not really happy with my solutions for some of these. In particular, I should be able to evaluate A5 without infinite series; and I should be able to do A2 directly without induction. If I have the energy to improve these solutions, you may check back at this site for updated solutions.

Problem A1

We call a finite set of positive integers *feasible* if every member of S has the form 2^r3^s, and no member of S divides any other member of S. We call a positive integer n *feasible* if $n = \sum S$ for some feasible set S. We are required to show that every positive integer is feasible. If not, then there exists a smallest infeasible number n. If n is even, say $n = 2k$ for some positive integer $k < n$ then we have $k = \sum S$ for some feasible set S and $n = \sum 2S$ where the set $2S = \{2a : a \in S\}$ is clearly feasible, contrary to assumption. Otherwise n is odd. Let $s \geq 0$ be maximal such that $3^s \leq n$, so that $3^s > \frac{1}{3}n$ and $n - 3^s = 2k$ for some positive integer k. We have $k = \sum S$ for some feasible set S, and since $k < \frac{1}{3}n < 3^s$, no member of S (or of $2S$) is divisible by 3^s. In particular $3^s \notin 2S$ and $n = \sum S'$ where $S' = \{3^s\} \cup 2S$. To check that S' is feasible, it remains only to observe that since 3^s is odd, it is not divisible by any member of $2S$. Thus n is feasible, a contradiction.

Problem A2

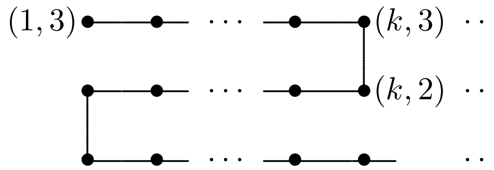
Let r_n be the number of rook tours of $S = S_n = \{1, 2, \ldots, n\} \times \{1, 2, 3\}$, and let s_n be the number of 'alternate rook tours' of S starting at $(1, 3)$ and ending at $(n, 1)$. Every rook tour uniquely determines an integer $k \in \{1, 2, \ldots, n\}$ such that the first vertical segment in the tour is from $(k, 1)$ to $(k, 2)$ as shown:



The right-hand portion of this tour not shown consists of an alternate rook tour of $\{k{+}1, k{+}2, \ldots, n\} \times \{1, 2, 3\}$, which shows that for $n \geq 2$,

$$r_n = s_1 + s_2 + \cdots + s_{n-1}\,.$$

Similarly, each alternate rook tour uniquely determines an integer $k \in \{1, 2, \ldots, n\}$ such that the first vertical segment in the tour is from $(k, 3)$ to $(k, 2)$ as shown:



The right-hand portion of this tour not shown consists of a rook tour of $\{k+1, k+2, \ldots, n\} \times \{1, 2, 3\}$, which shows that for $n \geq 2$,

$$s_n = 1 + r_1 + r_2 + \cdots + r_{n-1} .$$

It is straightforward to check that

$$\begin{aligned} &r_1 = 0, \quad s_1 = 1; \\ &r_n = s_n = 2^{n-2} \quad \text{for } n \geq 2 \end{aligned}$$

where the case $n = 1$ is found directly, and the cases $n \geq 2$ are verified by induction.

Problem A3

(We assume that either n is even or that z is restricted to aa domain on which an analytic branch of $z^{n/2}$ may be defined.) Write $p(z) = \prod_{j=1}^{n}(z - a_j)$ where the zeroes $a_j \in \mathbb{C}$ have modulus 1. Logarithmic differentiation of $g(z) = p(z)/z^{n/2}$ yields

$$\frac{g'(z)}{g(z)} = \sum_{j=1}^{n} \frac{1}{z - a_j} - \frac{n}{2z}$$

and so

$$\frac{2zg'(z)}{g(z)} = \sum_{j=1}^{n} \frac{z + a_j}{z - a_j} .$$

If $|a| = 1$ then

$$2Re\Big(\frac{z + a}{z - a}\Big) = \frac{z + a}{z - a} + \frac{\overline{z} + \overline{a}}{\overline{z} - \overline{a}} = 2\frac{|z|^2 - 1}{|z - a|^2}$$

and so

$$2Re\frac{zg'(z)}{g(z)} = (|z|^2 - 1)\sum_{j=1}^{n} \frac{1}{|z - a_j|^2} .$$

If $g'(z) = 0$ then we must have $|z| = 1$ since the latter sum is positive.

Problem A4

Since $H^T H = nI$, the matrix $n^{-1/2}H$ is orthogonal; thus $\|Hv\| = n^{1/2}\|v\|$ for every $v \in \mathbb{R}^n$. By the Cauchy-Schwartz inequality,

$$|u^T Hv| \le \|u\|{\cdot}\|Hv\| = n^{1/2}\|u\|{\cdot}\|v\|.$$

for all $u, v \in \mathbb{R}^n$. Now suppose the upper left $a \times b$ submatrix of H has only 1's as entries. Take $u \in \mathbb{R}^n$ to have its first a entries equal to 1, and the remaining entries 0. Similarly $v \in \mathbb{R}^n$ has its first b entries equal to 1, and the remaining entries 0. Then

$$ab = |u^T Hv| \le n^{1/2}\|u\|{\cdot}\|v\| = n^{1/2}a^{1/2}b^{1/2}.$$

This implies that $ab \le n$.

Problem A5

(*Note: I would need to justify my term-by-term integration here. It's messy enough though, so I would rather look for a slicker proof...*) Denote by I the integral

$$I = \int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx = \int_0^1 \left(\sum_{n\ge1} \frac{(-1)^{n-1}}{n}\,x^n\right)\frac{dx}{x^2+1} = \sum_{n\ge1}(-1)^{n-1}I_n$$

where

$$I_n = \int_0^1 \frac{x^n\,dx}{x^2+1}\,.$$

For all $n \ge 0$ we have

$$I_{n+2} + I_n = \int_0^1 \frac{x^{n+2}+x^n}{x^2+1}\,dx = \int_0^1 x^n\,dx = \frac{1}{n+1}\,.$$

It follows easily by induction that

$$I_{2k} = (-1)^k\Big[\frac{\pi}{4} + \sum_{i=1}^k \frac{(-1)^i}{2i-1}\Big] \quad\text{and}\quad I_{2k+1} = (-1)^k\Big[\frac{\ln 2}{2} + \sum_{i=1}^k \frac{(-1)^i}{2i}\Big]$$

for all $k \ge 0$. Thus

$$\begin{aligned} I &= \sum_{k\ge1}\frac{1}{2k}I_{2k} - \sum_{k\ge0}\frac{1}{2k+1}I_{2k+1} \\ &= \sum_{k\ge1}\frac{(-1)^k}{2k}\Big[\frac{\pi}{4} + \sum_{i=1}^k \frac{(-1)^i}{2i-1}\Big] - \sum_{k\ge0}\frac{(-1)^k}{2k+1}\Big[\frac{\ln 2}{2} + \sum_{i=1}^k \frac{(-1)^i}{2i}\Big] \\ &= \frac{\ln 2}{2}{\cdot}\frac{\pi}{4} + \sum_{(i,k)\,:\,1\le i\le k}\frac{(-1)^{i+k}}{2k(2i-1)} - \frac{\pi}{4}{\cdot}\frac{\ln 2}{2} - \sum_{(i,k)\,:\,1\le i\le k}\frac{(-1)^{i+k}}{(2k+1)2i}\,. \end{aligned}$$

After cancelling the two constant terms we are left with two sums. In the first such sum, replace i by $j+1$. In the second sum, rename k as j, then rename i as k. This gives

$$\begin{aligned} I &= \sum_{(j,k)\,:\,0\le j<k} \frac{(-1)^{j+k-1}}{(2j{+}1)2k} + \sum_{(j,k)\,:\,1\le k\le j} \frac{(-1)^{j+k-1}}{(2j{+}1)2k} \\ &= \Bigl[\sum_{j\ge 0} \frac{(-1)^j}{2j{+}1}\Bigr]\Bigl[\sum_{k\ge 1} \frac{(-1)^{k-1}}{2k}\Bigr] = \frac{\pi}{4}\cdot\frac{\ln 2}{2} = \frac{\pi \ln 2}{8}\,. \end{aligned}$$

Problem A6

For each $i = 1, 2, \ldots, n$, let E_i be the event that the angle at P_i is acute. Note that E_i is actually the event that there exists a diameter of the circle with P_i on one side, and all other P_j's on the other side. We may coordinatise P_i by the angle $\theta_i \in [0, 2\pi)$ measured counterclockwise from P_1, so that $\theta_1 = 0$; and $\theta_2, \ldots, \theta_n$ are independent random variables uniformly distributed in $[0, 2\pi)$. Realising E_1 as a union of mutually exclusive possibilities gives the probability of E_1 as

$$\begin{aligned} Pr(E_1) &= Pr\bigl[\theta_2, \ldots, \theta_n \in (\pi, 2\pi)\bigr] \\ &\qquad + \sum_{i=2}^{n} Pr\bigl[\theta_i \in (0,\pi) \text{ and } \theta_j \in (\theta_i, \theta_i{+}\pi) \text{ for all } j \notin \{1, i\}\bigr] \\ &= \frac{1}{2^{n-1}} + (n{-}1)\cdot\frac{1}{2}\cdot\frac{1}{2^{n-2}} \\ &= \frac{n}{2^{n-1}}. \end{aligned}$$

Similarly $Pr(E_i) = n/2^{n-1}$ for all $i \in \{1, 2, \ldots, n\}$.

For any events E and F, we denote by $E \wedge F$ the event that both E and F occur; and by $E \vee F$ the event that E or F occurs (possibly both). We denote the conditional probability of E given F as $Pr(E \mid F)$. We next determine $Pr(E_1 \wedge E_2)$, the probability that the angles at P_1 and P_2 are both acute. Since θ_2 is equally likely to be in $(0, \pi)$ or in $(\pi, 2\pi)$, and by symmetry the conditional probability of $E_1 \wedge E_2$ given that $\theta_2 \in (0, \pi)$ equals the conditional probability of $E_1 \wedge E_2$ given that $\theta_2 \in (\pi, 2\pi)$, we have

$$\begin{aligned} Pr(E_1 \wedge E_2) &= Pr\bigl[E_1 \wedge E_2 \mid \theta_2 \in (0,\pi)\bigr] \\ &= Pr\bigl[\theta_3, \theta_4, \ldots, \theta_n \in (0, \theta_2) \mid \theta_2 \in (0,\pi)\bigr] \\ &\qquad + Pr\bigl[\theta_3, \theta_4, \ldots, \theta_n \in (\pi, \theta_2{+}\pi) \mid \theta_2 \in (0,\pi)\bigr] \\ &= 2Pr\bigl[\theta_3, \theta_4, \ldots, \theta_n \in (0, \theta_2) \mid \theta_2 \in (0,\pi)\bigr] \\ &= 2\int_0^{\pi} Pr\bigl[\theta_3, \theta_4, \ldots, \theta_n \in (0, \theta_2) \mid \theta_2{=}\theta\bigr]\,\frac{d\theta}{\pi} \\ &= 2\int_0^{\pi} \left(\frac{\theta}{2\pi}\right)^{n-2} \frac{d\theta}{\pi} \\ &= \frac{1}{(n{-}1)2^{n-1}}. \end{aligned}$$

Similarly $Pr(E_i \wedge E_j) = 1/(n-1)2^{n-1}$ whenever $i \neq j$. For $n \geq 4$, it is not possible for more than two of the angles to be acute. For suppose that the angles at P_1, P_2 are acute. We may suppose that $\theta_2 \in (0, \pi)$; then as above we have $\theta_3, \dots, \theta_n$ all lie in the interval $(0, \theta_2)$, or else all lie in $(\pi, \theta_2+\pi)$; and then it is easy to see that the angles at $P_3, P_4, \dots, P_n$ are obtuse.

By inclusion-exclusion we obtain the probability that at least one of the angles at $P_1, P_2, \dots, P_n$ is acute:

$$\begin{aligned} Pr(E_1 \vee E_2 \vee \cdots \vee E_n) &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} Pr(E_{i_1} \wedge E_{i_2} \wedge \cdots \wedge E_{i_k}) \\ &= nPr(E_1) - \frac{n(n-1)}{2} Pr(E_1 \wedge E_2) \\ &= n \cdot \frac{n}{2^{n-1}} - \frac{n(n-1)}{2} \cdot \frac{1}{(n-1)2^{n-1}} \\ &= \frac{n(n-2)}{2^{n-1}}. \end{aligned}$$

Problem B1

The polynomial $P(x, y) = (y - 2x)(y - 2x - 1)$ satisfies the required condition. To see this, let $k = \lfloor a \rfloor$; then either

$$\begin{aligned} k \leq a < a+\tfrac{1}{2} \quad &\text{and} \quad \lfloor 2a \rfloor = 2k, \quad \text{or} \\ k+\tfrac{1}{2} \leq a < k+1 \quad &\text{and} \quad \lfloor 2a \rfloor = 2k+1. \end{aligned}$$

Problem B2

The Cauchy-Schwartz inequality gives

$$n = \sum_{i=1}^{n} k_i^{1/2} k_i^{-1/2} \leq \left(\sum_{i=1}^{n} k_i \right)^{1/2} \left(\sum_{i=1}^{n} k_i^{-1} \right)^{1/2} = \sqrt{5n-4}$$

and so $n^2 \leq 5n - 4$, i.e. $(n-1)(n-4) \leq 0$. This gives $n \in \{1, 2, 3, 4\}$. Since each $k_i \in \{1, 2, \dots, n\}$ we have only a short list of candidates for $(n; k_1, k_2, \dots, k_n)$. We check that all solutions are given by $(1; 1)$, $(3; 2, 3, 6)$ and $(4; 1, 1, 1, 1)$.

Problem B3

Since

$$\frac{d}{dx}\left[f(x) f\left(\frac{a}{x}\right) \right] = f'(x) f\left(\frac{a}{x}\right) - \frac{a}{x^2} f(x) f'\left(\frac{a}{x}\right) = \frac{a}{x} - \frac{a}{x^2} x = 0,$$

we see that $f(x)f(a/x) = c$ is a positive constant. Combining this with the relation $f'(x)f(a/x) = a/x$ gives $xf'(x) = kf(x)$ where $k = a/c > 0$ and so

$$\frac{d}{dx}\ln\bigl[x^{-k}f(x)\bigr] \;=\; \frac{d}{dx}\bigl[\ln f(x) - k\ln x\bigr] \;=\; \frac{f'(x)}{f(x)} - \frac{k}{x} \;=\; 0.$$

This shows that $x^{-k}f(x) = m$ is a positive constant, i.e. $f(x) = mx^k$. The relation $m = \left(ka^{k-1}\right)^{-1/2}$ then follows from $f'(a/x) = x/f(x)$ and the requirement that $m > 0$. Conversely, every function of this form is a solution.

Problem B4

Note that

$$\left(\frac{1+x}{1-x}\right)^n = (1+2x+2x^2+2x^3+2x^4+\cdots)^n = \left(\sum_{r\in\mathbb{Z}} x^{|r|}\right)^n = \sum_{m\geq 0} g(m,n)x^m$$

where $g(m,n)$ is the number of n-tuples of integers $(r_1, r_2, \ldots, r_n)$ such that $|r_1| + |r_2| + \cdots + |r_n| = m$. Divide both sides by $1-x$ to obtain

$$\begin{aligned}\frac{(1+x)^n}{(1-x)^{n+1}} &= \sum_{m\geq 0} g(m,n)(x^m + x^{m+1} + x^{m+2} + x^{m+3} + \cdots) \\ &= \sum_{m\geq 0} \bigl(g(0,n) + g(1,n) + g(2,n) + \cdots + g(m,n)\bigr)x^m = \sum_{m\geq 0} f(m,n)x^m.\end{aligned}$$

Multiply both sides by y^n and sum over $n \geq 0$ to obtain

$$\sum_{m,n\geq 0} f(m,n)x^m y^n = \frac{1}{1-x}\sum_{n\geq 0}\left(\frac{1+x}{1-x}\,y\right)^n = \frac{1}{1-x}\cdot\frac{1}{1-\left(\frac{1+x}{1-x}\right)y} = \frac{1}{1-x-y-xy}\,.$$

This rational function of x and y is symmetric in x and y, so we must have $f(m,n) = f(n,m)$ for all $m,n \geq 0$.

Problem B5

Suppose $P = P(x_1, \ldots, x_n)$ is not identically zero. We may suppose P is homogeneous, since every homogeneous component of P satisfies both (a) and (b). Abbreviate $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and $S = S(x_1, \ldots, x_n) = \sum_{i=1}^n x_i^2$. Let $r \geq 1$ be the integer uniquely determined by $P = S^r Q$ where the polynomial $Q = Q(x_1, \ldots, x_n)$ is not divisible by S. Then Q is also homogeneous, say of degree $k \geq 0$. Now

$$\frac{\partial^2 P}{\partial x_i^2} \;=\; 4r(r-1)x_i^2 S^{r-2}Q \;+\; 2rS^{r-1}Q \;+\; 4rS^{r-1}x_i\frac{\partial Q}{\partial x_i} \;+\; S^r\frac{\partial^2 Q}{\partial x_i^2}$$

and summing over i gives

$$0 \;=\; \Delta P \;=\; 4r(r{-}1)S^{r-1}Q \;+\; 2nrS^{r-1}Q \;+\; 4rS^{r-1}\sum_{i=1}^{n} x_i\frac{\partial Q}{\partial x_i} \;+\; S^r\Delta Q.$$

Since Q is homogeneous of degree k, Euler's identity gives $\sum_{i=1}^{n} x_i\frac{\partial Q}{\partial x_i} = kQ$. Thus

$$0 \;=\; 2r(2r{-}2{+}n{+}k)S^{r-1}Q \;+\; S^r\Delta Q.$$

Multiplying by S yields

$$0 \;=\; 2r(2r{-}2{+}n{+}k)P \;+\; S^{r+1}\Delta Q$$

where the constant $2r(2r{-}2{+}n{+}k)P$ is positive, so that S^{r+1} divides P, a contradiction.

Problem B6

Let A be the $n \times n$ matrix with all diagonal entries equal to t, and all off-diagonal entries equal to 1. Note that $v = (1, 1, \ldots, 1) \in \mathbb{R}^n$ is an eigenvector of A with eigenvalue $t{+}n{-}1$, and the orthogonal complement of v is an eigenspace of dimension $n{-}1$ with eigenvalue $t{-}1$. Expanding the determinant of A gives

$$\sum_{\pi\in S_n} \sigma(\pi)t^{\nu(\pi)} = (t{-}1)^{n-1}(t{+}n{-}1) = (t{-}1)^n + n(t{-}1)^{n-1}.$$

Integrating from $t = 0$ to $t = 1$ gives the required result.