Group Cohomology

Homology and cohomology groups are a very important tool in classifying extensions. The vague term ‘extensions’ is intended to include various kinds of objects: topological, geometric and algebraic. Here we explain how cohomology groups are useful in classifying central extensions of groups.

1. Modules

Let $G$ be a multiplicative group, and let $A$ be a $G$-module. This means that $A$ is an additive abelian group and that $G$ acts on $A$. We denote the image of an element $a \in A$ under an element $g \in G$ by $ga \in A$. To say that $G$ acts on $A$ means that

$$g(a + b) = ga + gb; \quad (gh)a = g(ha)$$

for all $a, b \in A; \ g, h \in G$.

1.1 Example: Linear Groups

We may take $A$ to be a vector space and let $G = GL(A)$, the group of all invertible linear transformations $A \to A$. Or we may take $G$ to be an arbitrary subgroup of $GL(A)$.

1.2 Example: Trivial Action

Take $A$ to be an arbitrary additive abelian group, and $G$ an arbitrary multiplicative group. The trivial action of $G$ on $A$ is defined by

$$ga = a$$

for all $a \in A, g \in G$.

2. Definition of Group Cohomology

Let $A$ be a $G$-module, as in Section 1. Denote by $C^k = C^k(G; A)$ the additive group consisting of all maps $\phi : G^{k+1} \to A$ such that

$$\phi(gg_0, gg_1, \ldots, gg_k) = g\phi(g_0, g_1, \ldots, g_k)$$
for all $g_0, g_1, \ldots, g_k, g \in G$. Such maps are called $k$-cochains. The coboundary of such a map $\phi \in C^k$ is the $(k+1)$-cochain $\delta \phi \in C^{k+1}$ defined by

$$(\delta \phi)(g_0, g_1, \ldots, g_{k+1}) = \sum_{0 \leq i \leq k+1} (-1)^i \phi(g_0, g_1, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{k+1}).$$

It is easy to check that $\delta \phi$ satisfies the condition above to be a cochain; also that $\delta^2 = 0$ so that we have a cochain complex

$$\cdots \leftarrow C^3 \leftarrow C^2 \leftarrow C^1 \leftarrow C^0 \leftarrow 0.$$

As usual we define the $k$-th cohomology group of this complex by

$$H^k(G; A) = Z^k(G; A)/B^k(G; A)$$

where $Z^k(G; A)$ is the kernel of $\delta : C^k \to C^{k+1}$ (the additive group of cocycles) and $B^k(G; A)$ is the image of $\delta : C^{k-1} \to C^k$ (the additive group of coboundaries).

The preceding description of cochains can be abbreviated by a process of de-homogenization as we now explain. Every cochain $\phi$ as above gives rise to a map $f : G^k \to A$ defined by

$$f(g_1, g_2, \ldots, g_k) = \phi(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_k).$$

Conversely we may recover $\phi$ from $f$ via

$$\phi(g_0, g_1, g_2, \ldots, g_k) = g_0 f(g_0^{-1}g_1^{-1}g_2, \ldots, g_{k-1}^{-1}g_k).$$

Using this bijection $\phi \leftrightarrow f$ we may identify $C^k(G; A)$ with the additive group of all functions $G^k \to A$. (For those who are familiar with homogeneous coordinates for projective space, this is analogous to the one-to-one correspondence between homogeneous and nonhomogeneous coordinates for points in projective space.) Now we may express the coboundary operator in our new notation as

$$(\delta f)(g_1, g_2, \ldots, g_k) = g_1 f(g_2, g_3, \ldots, g_{k+1}) - f(g_1g_2, g_3, \ldots, g_{k+1}) + f(g_1, g_2g_3, g_4, \ldots, g_{k+1}) - \cdots + (-1)^{k+1} f(g_1, g_2, \ldots, g_k).$$

It is this expression for the coboundary operator, rather than the previous, that we shall use in practice. The only point of giving the previous description in terms of $\phi$ is to motivate this unusual-looking formula.
2.1 Example: $k = 0$

A 0-cochain is a function $G^0 \to A$. Such a function has no arguments, and so it is really a constant $a \in A$. The coboundary of such a constant is the map

$$\delta a : G \to A, \quad g \mapsto ga - a.$$ 

The 0-cocycles are the elements $a \in A$ that are fixed by every element of $G$.

2.2 Example: $k = 1$

A 1-cochain is a function $f : G \to A$. Its coboundary is the map

$$\delta f : G^2 \to A, \quad (\delta f)(g, h) = gf(h) - f(gh) + f(g).$$

The 1-cocycles are functions $f : G \to A$ satisfying

$$f(gh) = f(g) + gf(h).$$

Such maps are called crossed homomorphisms or derivations. Note that if the action of $G$ on $A$ is trivial, these are the same as homomorphisms $G \to A$. As a special case one checks directly that every 1-coboundary is a 1-cocycle. Such maps have the form $f(g) = ga - a$ where $a \in A$ is fixed, and are called principal crossed homomorphisms or inner derivations. They satisfy

$$f(g) + gf(h) = (ga - a) + g(ha - a) = ga - a + gha - ga = gha - a = f(gh)$$

as required.

2.3 Example: $k = 2$

A 2-cochain is a function $f : G^2 \to A$. Its coboundary is the map $\delta f : G^3 \to A$ defined by

$$(\delta f)(g, h, \ell) = gf(h, \ell) - f(gh, \ell) + f(g, h\ell) - f(g, h).$$

Amazingly, such expressions arise naturally in the study of group extensions.
3. Products of Groups

Products (whether direct or semidirect) can be constructed either internally or externally. To motivate the distinction, consider the probably more familiar situation of direct sums in linear algebra.

3.1 Sums of Vector Spaces

Given two vector spaces $U$ and $W$ (over the same field $F$) one may construct their (external) direct sum which is the new vector space

$$ V = U \oplus W = \{(u, w) : u \in U, w \in W\} $$

with componentwise addition and scalar multiplication defined by

$$(u, w) + (u', w') = (u+u', w+w'), \quad c(u, w) = (cu, cw)$$

for all $u, u' \in U; w, w' \in W; c \in F$. Alternatively, given a vector space $V$ and two subspaces $U, W \subseteq V$, we can realize $V$ as the (internal) direct sum of $U$ and $W$, denoted again as $V = U \oplus W$, provided $U \cap W = \{0\}$ and $U + W = V$. The latter condition means that every vector $v \in V$ can be expressed as $v = u + w$ for some $u \in U$ and $w \in W$; and the preceding condition means that such $u$ and $w$ are uniquely determined by $v$.

Abstractly there is no distinction between internal and internal direct sums. The difference is only in presentation: namely, does one first define $U$ and $W$, then construct $V$ as their direct sum? or does one first construct $V$ and then identify a pair of complementary subspaces $U, W \subseteq V$?

Having said this, there is however one subtle distinction between the use of the notation ‘$\oplus$’ for internal and external and internal direct sums, as shown by the following example. We may construct the two-dimensional real vector space $\mathbb{R}^2$ as simply the external direct sum $\mathbb{R} \oplus \mathbb{R}$. However when we view the vector space $\mathbb{R}^2$ as an internal direct sum of two one-dimensional subspaces $U$ and $W$, these two subspaces should be disjoint. How can we then have $U = W = \mathbb{R}$? This confusion is resolved by observing that $U$ and $W$ are distinct one-dimensional subspaces, namely $U = \{(x, 0) : x \in \mathbb{R}\}$ (the $x$-axis) and $W = \{(0, y) : y \in \mathbb{R}\}$ (the $y$-axis). In this case we may rather say $U \cong W \cong \mathbb{R}$ to avoid the notational confusion just observed.

3.2 Direct Products of Groups

We generalize the previous section by taking $H$ and $K$ to be two groups. We assume for now that both $H$ and $K$ are multiplicative. The (internal) direct product of $H$ and $K$ is the group

$$ G = H \times K = \{(h, k) : h \in H, k \in K\} $$
with componentwise multiplication

$$(h, k)(h', k') = (hh', kk').$$

Note that we may identify $H$ with the subgroup $\{(h, 1) : h \in H\}$, and identify $K$ with the subgroup $\{(1, k) : k \in K\}$. With this identification, we observe that the subgroups $H$ and $K$ are complementary, i.e. $G = HK = \{hk : h \in H, k \in K\}$ (recall the identification of $h$ with $(h, 1)$ and $k$ with $(1, k)$) and $H \cap K = 1$ (so that every $g \in G$ can be uniquely expressed as $g = hk$) for $h$ and $k$ as above. Moreover these two subgroups are normal and they commute with each other: $hk = kh$ for all $h \in H$ and $k \in K$.

Conversely, given a group $G$, in order to recognize $G$ as the direct product of two subgroups $H, K \leq G$, we require that $G = HK$, $H \cap K = 1$, and $H$ commutes with $K$ (in particular both $H$ and $K$ are normal subgroups). We then write $G = HK = H \times K$, the (internal) direct product of $H$ and $K$.

### 3.3 Semidirect Products of Groups

Here we generalize the notion of product even further. Let $H$ and $K$ be groups, and suppose that $K$ acts on $H$. This means that each $k \in K$ determines a map $H \to H$ denoted by $h \mapsto h^k$ such that

$$(h_1h_2)^k = h_1^kh_2^k; \quad h^{k_1k_2} = (h^{k_1})^{k_2}$$

for all $h, h_1, h_2 \in H; \ k, k_1, k_2 \in K$. (Thus we are given not only groups $H$ and $K$ but also a homomorphism $K \to \text{Aut}(H)$.) Define the (external) semidirect product of $H$ and $K$ as

$$G = H \rtimes K = \{(h, k) : h \in H, k \in K\}$$

where the product in $G$ is defined by

$$(h_1, k_1)(h_2, k_2) = (h_1^{k_2}h_2, k_1k_2)$$

for all $h_i \in H, \ k_i \in K$. If you have never done this before, you should check that this actually does define a group; most importantly, this product is associative. Again $\{(h, 1) : h \in H\}$ is a subgroup (actually a normal subgroup) which we identify with $H$; and $\{(1, k) : k \in K\}$ is a subgroup (although not in general normal) which we identify with $K$. Note that $H$ and $K$ do not typically commute with each other; indeed

$$(1, k)^{-1}(h, 1)(1, k) = (h^k, 1)$$

so that the original action of $K$ on $H$ which was given, is realized as the action by conjugation in the group $G$. It is important to realize that the data required to construct the
group $G$ includes not only the groups $H$ and $K$, but also the choice of action of $K$ on $H$. In particular if one chooses the trivial action, one obtains simply a direct product as a special case.

Reversing our viewpoint, suppose we are given a group $G$ and two subgroups $H, K \leq G$ such that $H$ is normal and every element $g \in G$ is uniquely expressible as $g = hk$ where $h \in H$, $k \in K$ (i.e. $G = HK$ with $H \cap K = 1$). Then $G$ is the (internal) semidirect product of $H$ and $K$.

As a special case suppose $A$ is a module for a group $K$. Then the elements of the semidirect product $A \rtimes K$ can naturally be denoted as matrices

$$
\begin{pmatrix}
  k & 0 \\
  a & 1
\end{pmatrix}, \quad k \in K, \ a \in A
$$

with the convention that

$$
\begin{pmatrix}
  k_1 & 0 \\
  a_1 & 1
\end{pmatrix}
\begin{pmatrix}
  k_2 & 0 \\
  a_2 & 1
\end{pmatrix} =
\begin{pmatrix}
  k_1k_2 & 0 \\
  a_1^{k_2}a_2 & 1
\end{pmatrix}.
$$

Thus for example if $K = GL(A)$ where $A$ is a $k$-dimensional vector space over a field $F$, then then $A \rtimes K$ is isomorphic to the group of all invertible $(k+1) \times (k+1)$ matrices over $F$ with last column equal to the transpose of $(0, 0, \ldots, 0, 1)$.

As another example, consider a cyclic group $H = \{1, x, x^2, \ldots, x^{n-1}\}$ of order $n$, and let $K = \{1, y\}$ be a group of order 2. Then any semidirect product of $H$ by $K$ is either a direct product (in which $x$ commutes with $y$) or a dihedral group (in which $x^y = y^{-1}xy = x^{-1}$).

4. Group Extensions

A group extension of $A$ by $G$ (also called a group extension of $G$ by $A$) is a short exact sequence of groups

$$1 \rightarrow A \rightarrow H \rightarrow G \rightarrow 1.$$ 

This means that $A$ can be identified with a subgroup of $H$ in such a way that $H/A \cong G$. We often say simply that the group $H$ is an extension of $A$ by $G$, if there is no ambiguity regarding the choice of homomorphisms. Such a sequence is called split if $H$ has a subgroup complementary to $A$. (If such a complementary subgroup exists, it would necessarily be isomorphic to $G$; and then $H$ would be isomorphic to a semidirect product $A \rtimes G$.) Two extensions

$$1 \rightarrow A \rightarrow H_1 \rightarrow G \rightarrow 1, \quad 1 \rightarrow A \rightarrow H_2 \rightarrow G \rightarrow 1$$
are equivalent if there exist isomorphisms $\alpha, \beta, \gamma$ which yield a commutative diagram

\[
\begin{array}{ccccccccc}
1 & \rightarrow & A & \rightarrow & H_1 & \rightarrow & G & \rightarrow & 1 \\
\downarrow & & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \downarrow \\
1 & \rightarrow & A & \rightarrow & H_2 & \rightarrow & G & \rightarrow & 1.
\end{array}
\]

(It is only necessary to assume that $\alpha$ and $\gamma$ are isomorphisms, and that $\beta$ is a homomorphism, since by the Five Lemma $\beta$ is forced to also be an isomorphism.) Note that $H_1 \cong H_2$ in the case of equivalent extensions. However, the converse fails: if both $H_1$ and $H_2$ are extensions of $K$ by $G$, then they need not be equivalent; even if $H_1 \cong H_2$, the extensions may not be equivalent.

Group extensions are classified using cohomology. Consider especially the case that $A$ is abelian, and identify $A$ with its image in $G$. The action of $h \in H$ on $A$ by conjugation only depends on the coset $hA \in H/A \cong G$, so this gives an action of $G$ on $A$; thus $A$ is a $G$-module. We may ask, given an action of $G$ on $A$, for a determination of the equivalence classes of extensions of $A$ by $G$. These extensions are naturally in one-to-one correspondence with the elements of $H^2 = H^2(G;A)$; and the identity element of this group $H^2$ corresponds to a split extension (i.e. the semidirect product $A \rtimes G$). Moreover in the special case of a split extension $H = A \rtimes G$, the conjugacy classes of subgroups complementary to $A$, are naturally in one-to-one correspondence with elements of $H^1 = H^1(G;A)$; and the obvious choice of complementary subgroup (namely $G$) corresponds with the identity element of $H^1$. 