



Math 5555

# Abstract Algebra II

Book 3

For each  $i \in \{1, 2, \dots, k\}$ , we solve  $n_i$  equations (one for each  $t \in \{1, \dots, n_i\}$ )

in  $n_i$  unknowns  $\lambda_{i,j}$ ,  $j \in \{1, 2, \dots, n_i\}$ .

In general the  $n_i$  symmetric polynomials  $\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t$ ,  $1 \leq t \leq n_i$ , can be re-expressed in terms of the elementary symmetric polynomials  $e_j = e_j(\lambda_{i,1}, \dots, \lambda_{i,n_i})$  which are the coefficients of

Use  
(Newton's identities)

$$(x + \lambda_{i,1})(x + \lambda_{i,2}) \dots (x + \lambda_{i,n_i}) = x^{n_i} + e_1 x^{n_i-1} + e_2 x^{n_i-2} + \dots + e_{n_i} x + e_{n_i}$$

i.e.  $e_0 = \lambda_{i,1} \lambda_{i,2} \dots \lambda_{i,n_i}$

$$e_2 = \sum \lambda_{i,r} \lambda_{i,s}$$

$$e_1 = \lambda_{i,1} + \dots + \lambda_{i,n_i}$$

We will show: if  $|G| = n$  then  $\mathbb{C}G \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C})$  (algebra isomorphism)

where  $k =$  number of conjugacy classes in  $G$ .

The center of  $R$  (semisimple algebra) is

$$Z(R) = \{z \in R : zx = xz \text{ for all } x \in R\}$$

$Z(R) \subseteq R$  is a subalgebra: a subspace which is also a subring.

$$M(n, \mathbb{C}) = \{n \times n \text{ complex matrices}\}$$

$$Z(M(n, \mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$$

$$\uparrow I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$$

$$Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) = Z\left(\begin{bmatrix} * & & 0 \\ * & \lambda_1 I_{n_1} & \\ 0 & & * \end{bmatrix}\right) = \left\{ \begin{bmatrix} \lambda_1 I_{n_1} & & 0 \\ & \lambda_2 I_{n_2} & \\ 0 & & \lambda_k I_{n_k} \end{bmatrix} : \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$$

$$\dim \bigoplus_{i=1}^k M(n_i, \mathbb{C}) = \sum_{i=1}^k n_i^2 ; \quad \dim \left( Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) \right) = k$$

$$\dim \mathbb{C}G = n = |G|$$

$$\dim \mathbb{Z}(CG) = k = \text{no. of conj. classes.}$$

Let  $K_1, \dots, K_k \subset G$  be the conj. classes i.e.  $G = K_1 \sqcup K_2 \sqcup \dots \sqcup K_k$

For  $1 \leq i \leq k$ , let  $z_i = \sum_{g \in K_i} g = \text{sum of elements in } K_i$

$$z_i \in \mathbb{Z}(CG) \text{ because } \begin{aligned} gz_i &= z_i g \\ gz_i g^{-1} &= z_i \end{aligned}$$

$$\mathbb{Z}(CG) = \left\{ a_1 z_1 + \dots + a_k z_k : a_i \in \mathbb{C} \right\}$$

$$\text{Given } z \in \mathbb{Z}(CG), \text{ say } z = \sum_{x \in G} a_x x \quad a_x \in \mathbb{C}$$

$$\text{For all } g \in G, \quad \begin{aligned} gz &= zg \\ gzg^{-1} &= z \end{aligned}$$

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$$\mathbb{Z}G = \text{integral group ring of } G = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{Z} \right\}$$

$$\mathbb{Q}G = \text{rational group algebra} \quad \dots \quad \mathbb{Q}$$

$$\mathbb{R}G = \text{real group algebra} \quad \dots \quad \mathbb{R}$$

$$FG = \text{group algebra of } G \text{ over } F \quad \dots \quad F$$

$$F[G] = FG \quad \text{when } G \text{ is a group.}$$

$F[x, y, z]$  = polynomial algebra in  $x, y, z$  with coefficients in  $F$  (infinite dimensional)  
 as distinguished from  $Fx + Fy + Fz = \langle x, y, z \rangle_F = \{a_x x + a_y y + a_z z : a_x, a_y, a_z \in F\}$   
 which is a 3-dimensional vector space

If  $R$  is an algebra over  $F$  and  $S \subseteq R$  (any subset) then  
 the centralizer of  $S$  in  $R$  is

$$C_R(S) = \{z \in R : zs = sz \text{ for all } s \in S\} \subseteq R \text{ subalgebra}$$

(Also called the commutant of  $S$  in  $R$ ).

$$C_R(R) = Z(R)$$

Schur's Lemma (late 19<sup>th</sup> century)

Let  $R$  be a semisimple algebra over  $\mathbb{C}$ , e.g.  $R = M(n, \mathbb{C})$ , or  $\bigoplus_{i=1}^k M(n_i, \mathbb{C})$ , or  $\mathbb{C}G$ ,  $1/G < \infty$ .

Let  $M, N$  be  $R$ -<sup>simple</sup> modules and  $\phi : M \rightarrow N$  a homomorphism i.e.  $\phi(rm + sm') = r\phi(m) + s\phi(m')$

For all  $r, s \in R$ ;  $m, m' \in M$ .

(i) If  $M \not\cong N$  as  $R$ -modules then  $\phi = 0$ .

There are no nonzero homomorphisms between simple modules.

(ii) If  $M \cong N$ , say  $M = N$ , then  $\phi = cI$  for some  $c \in \mathbb{C}$ .

Proof (i) If  $\phi \neq 0$  then there exists  $v_0 \in M$  such that  $\phi(v_0) \neq 0$ , so  $\underline{R\phi(v_0)} \subseteq \phi(M) \subseteq N$   
 is a nonzero submodule of  $N$ . Since  $N$  is simple,  $R\phi(v_0) = N$ .

The kernel of  $\phi : M \rightarrow N$  is a submodule of  $M$ .

Since  $M$  is simple  $\ker \phi = 0$  or  $M$ . But  $\phi \neq 0$  ( $\phi(v_0) \neq 0$ ) we have  
 $\ker \phi = 0$ . By the first isomorphism theorem,  $M \cong M / \underbrace{0}_{\ker \phi} \cong \phi(M) = N$

This contradicts  $M \not\cong N$ .

(ii) Let  $\phi: M \rightarrow M$  be a homomorphism of the simple  $R$ -module  $M$ .  
 In particular  $\phi$  is a  $\mathbb{C}$ -linear transformation of a finite dimensional complex vector space so there exists  $v_0 \in M$ ,  $v_0 \neq 0$  such that  $\phi(v_0) = cv_0$  for some  $c \in \mathbb{C}$  ( $\mathbb{C}$  is alg. closed). Let  $\tilde{\phi} = \phi - cI$  so  $\tilde{\phi}$  is a homomorphism of  $R$ -algebras:

$$\tilde{\phi}(rm) = \phi(rm) - cirm = r\phi(m) - crm = r(\phi(m) - cm) = r\tilde{\phi}(m)$$

$$\tilde{\phi}(m+m') = \tilde{\phi}(m) + \tilde{\phi}(m') \quad (r \in R; m, m' \in M)$$

$$v_0 \in \ker \tilde{\phi} \neq 0 \Rightarrow \ker \tilde{\phi} = M \Rightarrow \tilde{\phi} = 0 \Rightarrow \phi = cI. \quad \square$$

Remark If  $M \cong N$  as  $R$ -modules there is an isomorphism  $A: M \rightarrow N$   
 (A invertible  $n \times n$  matrix,  $n = \dim M = \dim N$ )  
 $A(rm) = rAm$  for all  $r \in R$ .

then the  $R$ -module homomorphisms  $M \rightarrow N$  all have the form  $cA$ ,  $c \in \mathbb{C}$ .

The choice of field  $\mathbb{C}$  is important for Schur's lemma e.g.

$$G = \{1, g, g^2, g^3\} \text{ cyclic of order 4, } \pi: G \rightarrow GL_2(\mathbb{R})$$

$$g \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$M = \mathbb{R}^2$  is an  $RG$ -module using  $\pi$

$$\pi(g) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$\pi$  makes  $M$  into an  $R$ -module for  $R = RG$

$M$  is a simple module but the  $R$ -homomorphisms  $M \rightarrow M$  are more than just  $aI = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ ; we have  $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$  ( $a, b \in \mathbb{R}$ )

Recall Schur's lemma:

If  $\pi: G \rightarrow GL_n(\mathbb{C})$  is an irreducible representation then the only  $n \times n$  matrices commuting with  $\pi(g)$ ,  $g \in G$  are the scalar matrices  $\lambda I_n$  ( $\lambda \in \mathbb{C}$ )

If  $\pi, \pi'$  are representations of,  $\pi: G \rightarrow GL_m(\mathbb{C})$ ,  $\pi': G \rightarrow GL_n(\mathbb{C})$  then  $\pi$  defines an action of  $G$  on  $\mathbb{C}^m$ .  $g \in G$  acts on  $v \in \mathbb{C}^m$  ( $m \times 1$  column vector) as  $gv = \pi(g)v \in \mathbb{C}^m$

and  $\pi'$  defines an action of  $G$  on  $\mathbb{C}^n$ ,  $gv = \pi'(g)v \in \mathbb{C}^n$

then a  $\mathbb{C}G$ -homomorphism of the corresponding modules is (concretely) an  $n \times m$  matrix  $A \in \mathbb{C}^{n \times m}: \mathbb{C}^m \rightarrow \mathbb{C}^n$  such that

$$\begin{array}{ccc} \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \\ \pi(g) \downarrow & & \downarrow \pi'(g) \\ \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \end{array} \text{ commutes for every } g \in G \text{ i.e. } \pi'(g)A = A\pi(g)$$

for all  $g \in G$

Such a matrix  $A \in \mathbb{C}^{n \times m}$  is an intertwining operator (a linear transformation which respects/preserves the action of  $G$ ).

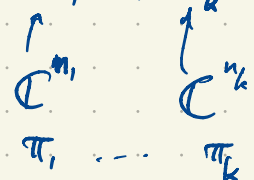
i.e.  $\alpha A = A \alpha$  for all  $\alpha \in \mathbb{C}G$ .

$\alpha(Av) = A(\alpha v)$   $A$  is a homomorphism of  $\mathbb{C}G$ -modules.

If  $V = \mathbb{C}^m$  and  $W = \mathbb{C}^n$  are irreducible (simple)  $\mathbb{C}G$ -modules ( $\pi, \pi'$  irred. representations):

- (i) inequivalent  $\Rightarrow A = 0$
- (ii)  $\pi \cong \pi'$  equivalent  $\Rightarrow A = \lambda \cdot$  fixed invertible  $n \times m$  matrix.

p.10 of handout. Theorem 4.4 (Wedderburn's Theorem)  
 Let  $R = \mathbb{C}G$ ,  $|G| < \infty$ . Let  $M_1, \dots, M_k$  be the distinct  $R$ -modules up to isomorphism.



The  $M_i$ -homogeneous part of  $R$  is the sum of all the left ideals of  $R$  isomorphic to  $M_i$ .

eg.  $R = M(3, \mathbb{C})$ . has only one simple module up to isomorphism,  $\mathbb{C}^3$ .

$A \in R$  acts on  $V = \mathbb{C}^3$  by  $\underbrace{v}_{3 \times 1} \mapsto \underbrace{Av}_{\substack{3 \times 3 \\ 3 \times 1}}$

$$R = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \quad \text{contains } \begin{bmatrix} a & b & 0 \\ b & b & 0 \\ c & c & 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Minimal Simple

Submodules (left ideals) of the regular representation are all isomorphic to  $\mathbb{C}^3$  as  $R$ -modules.

The  $\mathbb{C}^3$ -homogeneous part of this module is all of  $R$ .

$$R = \left[ \begin{array}{c|c} \boxed{1*} & \emptyset \\ \hline \emptyset & \boxed{***} \end{array} \right] \quad \text{Semisimple of dimension 4.}$$

$$\begin{aligned} \dim M_i &= n_i \\ \dim M_i(R) &= n_i^2 \end{aligned}$$

$R$  has three iso. types. of simple modules of dimension 1, 2, 3.

$$R = \underbrace{\begin{bmatrix} * & \emptyset \\ \emptyset & \emptyset \end{bmatrix}}_{M_1(R)} \oplus \underbrace{\begin{bmatrix} \emptyset & * \\ \emptyset & \emptyset \end{bmatrix}}_{M_2(R)} \oplus \underbrace{\begin{bmatrix} \emptyset & \emptyset \\ \emptyset & *** \end{bmatrix}}_{M_3(R)}$$

If  $n_i = \dim M_i$   
 then the  $M_i$ -homo. part of  $R = \mathbb{C}G$  is the sum of all submodules isomorphic to  $M_i$ .

$R$  semisimple algebra over  $\mathbb{C}$

$\text{End}_R(R) = \{ \underbrace{R\text{-endomorphisms of } R}_{\text{into itself}} \}$  is the set of maps  $\phi: R \rightarrow R$  such that  $\phi(rx) = r\phi(x)$  for all  $r, x \in R$

Lemma  $\text{End}_R(R)$  is anti-isomorphic to  $R$ .

For every  $a \in R$  we have  $\phi_a: R \rightarrow R, \phi_a(x) = xa$ .

$$\phi_a(rx) = rxa = r\phi_a(x) \Rightarrow \phi_a \in \text{End}_R(R).$$

The map  $R \rightarrow \text{End}_R(R), a \mapsto \phi_a \in \text{End}_R(R)$  is bijective.

If  $a \in R$  such that  $\phi_a = 0$  then  $\phi_a(1) = 0$  so  $a = 0$ . So  $a \mapsto \phi_a$  is one-to-one.

$$\phi_{ab}(r) = rab = \phi_b(ra) = \phi_b(\phi_a(r)) = \phi_b \circ \phi_a(r) \Rightarrow \phi_b \circ \phi_a = \phi_{ab}$$

If  $G$  is a group then  $a \mapsto a^{-1}$  is an anti-automorphism  $f(ab) = f(b)f(a)$   
Similarly with an algebra.

If  $R$  is an algebra then the opposite algebra  $R^\circ$  is the algebra with the same elements as  $R$  with same vector space structure; only the multiplication is replaced by  $x * y = \underbrace{yx}_{\text{in } R}$   $x(yz) = x * (y * z) = (zy)x = z(yx) = (x * y) * z$

$R^\circ$  is not necessarily isomorphic to  $R$ . It is anti-isomorphic.

$G^\circ \cong G$  because we can compose two anti-isomorphisms to get an isomorphism  
 $G \xrightarrow{\text{inverse}} G \rightarrow G^\circ$  If  $R$  is a division algebra then  $R^\circ \cong R$  in the same way.

What are the  $R$ -endomorphisms of  $R = \mathbb{C}G$ ?  $\text{End}_R(R) = R^o$ .

$R = I_1 \oplus I_2 \oplus \dots \oplus I_m$  as a direct sum of minimal left ideals.

If  $\phi \in \text{End}_R(R)$  then we can represent  $\phi$  using an  $m \times m$  matrix over  $R$ :

$$\phi(r) =$$

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \quad r_i \in I_i$$

$r = r_1 + \dots + r_m$  unique choice of  $r_i \in I_i$

$\phi(r) = \phi(r_1) + \dots + \phi(r_m)$ , decompose each term with respect to our decomposition

$$= \sum_{i=1}^m \left( \sum_{j=1}^m \phi_{ij} \cdot (r_j) \right)$$

$$\phi_{ij} \in \text{Hom}_R(I_i, I_j) = \begin{cases} 0 & \text{if } I_i \not\cong I_j \\ \mathbb{C}I_{n_i} & \text{if } I_i \cong I_j \end{cases}$$

Schur's Lemma

$\phi$  is represented by  $m \times m$  matrix over  $\mathbb{C}$

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1} & \dots & \dots & \phi_{mm} \end{bmatrix} = \begin{bmatrix} \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} & & & \\ & \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} & & \\ & & \dots & \\ & & & \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} \\ & & & & 0 \end{bmatrix}$$

$\in M(n_1, \mathbb{C})$        $\uparrow$        $M(n_2, \mathbb{C})$

$\text{End}_R(R)$  anti-iso. to  $\bigoplus_{i=1}^k M(n_i, \mathbb{C})$

$\parallel$   
 $R^o$  anti-iso  
 $= R$

$R = \mathbb{C}G$  has an anti-isomorphism  $\sum a_g g \mapsto \sum a_g g^{-1}$

$$R \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C})$$

p. 29 The center of the group algebra

$G$  finite group  $n = |G| = \dim \mathbb{C}G$

$$Z(\mathbb{C}G) = \{ \alpha \in \mathbb{C}G : \alpha \gamma = \gamma \alpha \text{ for all } \gamma \in \mathbb{C}G \} \subseteq \mathbb{C}G \text{ subalgebra}$$

$\dim Z(\mathbb{C}G) = k = \text{number of conjugacy classes of } G$

$K_1, K_2, \dots, K_k$  : conjugacy classes of  $G = K_1 \sqcup \dots \sqcup K_k$  partition

$$K_1 = \{1\}$$

$$\sum_{i=1}^k |K_i| = n = |G|$$

$$\text{Irr}(G) = \{ \chi_1, \chi_2, \dots, \chi_k \} \quad n_i = \chi_i(1) = \deg \chi_i$$

$$n_1^2 + n_2^2 + \dots + n_k^2 = n$$

Moreover  $n_i | n$  which we will prove today or Monday.

Irr. reps. of  $G$  are  $\pi_j : G \rightarrow \text{GL}_{n_j}(\mathbb{C})$

Basis for  $Z(\mathbb{C}G)$  is  $\{ \gamma_1, \gamma_2, \dots, \gamma_k \}$  where  $\gamma_i = \text{sum of elements in } K_i$ . Reps of  $K_1, \dots, K_k$  are  $\gamma_1, \dots, \gamma_k$

$$\mathbb{C}G \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C}) = \left[ \begin{array}{ccc} \begin{array}{c} n_1 \times n_1 \\ * \\ \end{array} & \dots & \begin{array}{c} n_k \times n_k \\ * \\ \end{array} \\ \begin{array}{c} n_2 \times n_2 \\ \end{array} & \dots & \begin{array}{c} n_k \times n_k \\ \end{array} \end{array} \right]$$

$$Z(\mathbb{C}G) \cong Z\left(\bigoplus M(n_i, \mathbb{C})\right) = \left\{ \left[ \begin{array}{ccc} \omega_1 I_{n_1} & & \\ & \omega_2 I_{n_2} & \\ & & \ddots \\ & & & \omega_k I_{n_k} \end{array} \right] : \omega_1, \dots, \omega_k \in \mathbb{C} \right\}$$

The iso.  $\mathbb{C}G \rightarrow \bigoplus_{i=1}^k M(n_i, \mathbb{C})$  is defined on our basis  $g \mapsto \left[ \begin{array}{ccc} \pi_1(g) & & \\ & \pi_2(g) & \\ & & \ddots \\ & & & \pi_k(g) \end{array} \right]$

$\pi_j: G \rightarrow GL_{n_j}(\mathbb{C})$  is a group homo. extending to algebra homo.

$\pi_j: \mathbb{C}G \rightarrow M(n_j, \mathbb{C})$ , defined by  $\pi_j(\sum_g a_g g) = \sum_g a_g \pi_j(g)$ .

Restrict to  $z \in Z(\mathbb{C}G)$ .  $zg = gz$  for all  $g \in G$

$\Rightarrow \pi_j(z) \pi_j(g) = \pi_j(zg) = \pi_j(gz) = \pi_j(g) \pi_j(z) \Rightarrow \pi_j(z) = c I_{n_j}$ , since  $c \in \mathbb{C}$

In particular  $\pi_j(\gamma_i) = \underbrace{\omega_j(\gamma_i)}_{\in \mathbb{C}} I_{n_j}$  since  $\gamma_i \in Z(\mathbb{C}G)$  by Schur's lemma.

Denote  $\omega(\gamma_i) = \omega_j(\gamma_i) = \omega_\chi(\gamma_i)$  where  $\chi = \chi_j = \text{tr } \pi_j$ .

$\pi_j(\gamma_i) = \omega_j(\gamma_i) I_{n_j}$  Take trace on both sides.  $\gamma_i = \sum_{g \in K_i} g$

$|K_i| \text{tr } \pi_j(g_i) = |K_i| \chi_j(g_i) = n_j \omega_j(\gamma_i) = \chi_j(1) \omega_j(\gamma_i)$

$\omega_\chi(\gamma_i) = \frac{|K_i| \chi(g_i)}{\chi(1)}$

(p. 30)

We will see that these values  $\omega_\chi(\gamma_i)$  are algebraic integers.

pos. integers

algebraic integers  
i.e. root of mono poly. with coeffs in  $\mathbb{Z}$ .

integers with

To show  $\frac{n}{n_i} \in \mathbb{Z}$ , show it's an algebraic integer and a rational number.

Use  $\{\text{alg. integers}\} \cap \mathbb{Q} = \mathbb{Z}$

$Z(CG)$  is an algebra. (subalgebra)

In particular  $\gamma_i \gamma_j \in Z(CG) \Rightarrow \gamma_i \gamma_j = \sum_{l=1}^k a_{ijl} \gamma_l$

$a_{ijk} \in \mathbb{Z}$

$a_{ijl}$  ( $i, j, l \in \{1, 2, \dots, k\}$ ) are the structure constants of  $Z(CG)$ .

Additively,  $Z(CG) \cong \mathbb{C}^k$   
 $\sum_{j=1}^k b_j \gamma_j$

$Z(\mathbb{Z}G) =$  algebra gen. by  $\gamma_1, \dots, \gamma_k$  over  $\mathbb{Z}$   
 $\{ \sum_j b_j \gamma_j : b_j \in \mathbb{Z} \}$

Additively:  $\mathbb{Z}^k$

The multiplicative structure is entirely described by the struct. constants  $a_{ijl}$ .

$a_{ijl}$  = ~~brunk~~ determined using the char. table of  $G$ .

Eg.  $G = S_3 = \underbrace{\{()\}}_{K_1} \sqcup \underbrace{\{(12), (13), (23)\}}_{K_2} \sqcup \underbrace{\{(123), (132)\}}_{K_3}$

In  $\mathbb{Z}G \subset CG$

$\gamma_1 = ()$ ,  $\gamma_2 = (12) + (13) + (23)$ ,  $\gamma_3 = (123) + (132)$

$\gamma_1 \gamma_2 = \gamma_2$ ,  $\gamma_1 \gamma_3 = \gamma_3$ ,  $\gamma_1 \gamma_1 = \gamma_1$

$\gamma_2 \gamma_2 = ((12) + (13) + (23))((12) + (13) + (23)) = 3() + 3(123) + 3(132) = 3\gamma_3$

$\gamma_3 \gamma_3 = ((123) + (132))((123) + (132)) = 2() + (123) + (132) = 2\gamma_1 + \gamma_3$

$\gamma_2 \gamma_3 = \gamma_3 \gamma_2 = ((12) + (13) + (23))((123) + (132)) = 2\gamma_2$

In a large group, these expansions look unmanageable this way but: we can compute  $a_{ijk}$  directly from the character table of  $G$ .

In  $S_3$ , each 3-cycle can be expressed as a product of two transpositions in exactly  $a_{223} = 3$  different ways.

$\pi(z) = \omega_\chi(z) I$  for all  $z \in Z(CG)$ , where  $\pi: G \rightarrow GL_n(\mathbb{C})$  irred. rep. with character  $\chi$   
 $\omega_\chi(\gamma_i) = \frac{|K_i| \chi(\gamma_i)}{\chi(1)}$   $\omega_\chi: Z(CG) \rightarrow \mathbb{C}$  is an algebra homo;  $\omega_\chi(zz') = \omega_\chi(z)\omega_\chi(z')$  for all  $z, z' \in Z(CG)$ .

Theorem The structure constants of  $Z(CG)$  are  $a_{ijl} = \frac{|K_i| |K_j|}{|G|} \sum_{\chi \in \text{Irr} G} \frac{\chi(\gamma_i) \chi(\gamma_j) \overline{\chi(\gamma_l)}}{\chi(1)}$  (non-negative integers)

Proof  $\gamma_i \gamma_j = \sum_{l=1}^k a_{ijl} \gamma_l$   $\omega_\chi(\gamma_i) = \frac{|K_i| \chi(\gamma_i)}{\chi(1)}$  Apply this homomorphism to  $Z(CG)$  on both sides:

$\frac{|K_i| \chi(\gamma_i)}{\chi(1)} \cdot \frac{|K_j| \chi(\gamma_j)}{\chi(1)} = \sum_{l=1}^k a_{ijl} \frac{|K_l| \chi(\gamma_l)}{\chi(1)}$  Now multiply both sides by  $\chi(\gamma_s) \overline{\chi(\gamma_s)}$  ( $s=1, 2, \dots, k$ ) to get

$$\begin{aligned} \sum_{\chi} \frac{|K_i| |K_j| \chi(\gamma_i) \chi(\gamma_j) \overline{\chi(\gamma_s)}}{\chi(1)} &= \sum_{\chi} \sum_{l=1}^k a_{ijl} |K_l| \chi(\gamma_l) \overline{\chi(\gamma_s)} \\ &= \sum_{l=1}^k a_{ijl} |K_l| \underbrace{\sum_{\chi} \chi(\gamma_l) \overline{\chi(\gamma_s)}}_{\delta_{ls} \frac{|G|}{|K_l|}} \\ &= |G| a_{ijs} \quad \square \end{aligned}$$

Sum over  $\chi \in \text{Irr}(G)$  and use col. orthogonality of char table of  $G$

$$\begin{aligned} \sum_{\chi \in \text{Irr} G} \chi(\gamma_l) \overline{\chi(\gamma_s)} &= \begin{cases} 0, & \text{if } l \neq s \\ |C_G(\gamma_l)|, & \text{if } l = s \end{cases} \\ &= \frac{|G|}{|K_l|} \end{aligned}$$

So why are  $\omega_\chi(\gamma_i) \in \mathbb{Z}$ ?

$$\gamma_i \gamma_j = \sum_{\ell=1}^k a_{ij\ell} \gamma_\ell$$

$$\omega_\chi(\gamma_i) \omega_\chi(\gamma_j) = \sum_{\ell=1}^k a_{ij\ell} \omega_\chi(\gamma_\ell)$$

In  $Z(CG)$ , we consider left-multiplication by  $\omega_\chi(\gamma_i)$  as a linear transformation with  $k \times k$  matrix (fixed:)

$$A_i = A = [a_{ij\ell}]_{j,\ell=1,\dots,k}$$

If  $v = \begin{bmatrix} \omega_\chi(\gamma_1) \\ \vdots \\ \omega_\chi(\gamma_k) \end{bmatrix}$  then  $Av = \omega_\chi(\gamma_i)v$  so  $v$  is an eigenvector with eigenvalue  $\omega_\chi(\gamma_i)$ .

Check  $v \neq 0$ : on Wednesday.