



Math 5555

Abstract Algebra II

Book 3

For each $i \in \{1, 2, \dots, k\}$, we solve n_i equations (one for each $t \in \{1, \dots, n_i\}$)

in n_i unknowns $\lambda_{i,j}$, $j \in \{1, 2, \dots, n_i\}$.

In general the n_i symmetric polynomials $\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t$, $1 \leq t \leq n_i$, can be re-expressed in terms of the elementary symmetric polynomials $e_j = e_j(\lambda_{i,1}, \dots, \lambda_{i,n_i})$ which are the coefficients of

Use
(Newton's identities)

$$(x + \lambda_{i,1})(x + \lambda_{i,2}) \dots (x + \lambda_{i,n_i}) = x^{n_i} + e_1 x^{n_i-1} + e_2 x^{n_i-2} + \dots + e_{n_i} x + e_{n_i}$$

i.e. $e_n = \lambda_{i,1} \lambda_{i,2} \dots \lambda_{i,n_i}$

$$e_2 = \sum \lambda_{i,r} \lambda_{i,s}$$

$$e_1 = \lambda_{i,1} + \dots + \lambda_{i,n_i}$$

We will show: if $|G| = n$ then $\mathbb{C}G \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C})$ (algebra isomorphism)

where $k =$ number of conjugacy classes in G .

The center of R (semisimple algebra) is

$$Z(R) = \{z \in R : zx = xz \text{ for all } x \in R\}$$

$$M(n, \mathbb{C}) = \{n \times n \text{ complex matrices}\}$$

$Z(R) \subseteq R$ is a subalgebra: a subspace which is also a subring.

$$Z(M(n, \mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$$

$$\uparrow I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$$

$$Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) = Z\left(\begin{bmatrix} * & & 0 \\ * & * & \\ 0 & & * \end{bmatrix}\right) = \left\{ \begin{bmatrix} \lambda_1 I_{n_1} & & 0 \\ & \lambda_2 I_{n_2} & \\ 0 & & \lambda_k I_{n_k} \end{bmatrix} : \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$$

$$\dim \bigoplus_{i=1}^k M(n_i, \mathbb{C}) = \sum_{i=1}^k n_i^2 ; \quad \dim \left(Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) \right) = k$$

$$\dim \mathbb{C}G = n = |G|$$

$$\dim \mathbb{Z}(CG) = k = \text{no. of conj. classes.}$$

Let $K_1, \dots, K_k \subset G$ be the conj. classes i.e. $G = K_1 \sqcup K_2 \sqcup \dots \sqcup K_k$

For $1 \leq i \leq k$, let $z_i = \sum_{g \in K_i} g = \text{sum of elements in } K_i$

$$z_i \in \mathbb{Z}(CG) \text{ because } \begin{aligned} gz_i &= z_i g \\ gz_i g^{-1} &= z_i \end{aligned}$$

$$\mathbb{Z}(CG) = \left\{ a_1 z_1 + \dots + a_k z_k : a_i \in \mathbb{C} \right\}$$

Given $z \in \mathbb{Z}(CG)$, say $z = \sum_{x \in G} a_x x \quad a_x \in \mathbb{C}$

For all $g \in G$, $\begin{aligned} gz &= zg \\ gzg^{-1} &= z \end{aligned}$

$$\mathbb{Z}G = \text{integral group ring of } G = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{Z} \right\}$$

$$\mathbb{Q}G = \text{rational group algebra} \quad \dots \quad \mathbb{Q}$$

$$\mathbb{R}G = \text{real group algebra} \quad \dots \quad \mathbb{R}$$

$$FG = \text{group algebra of } G \text{ over } F \quad \dots \quad F$$

$$F[G] = FG \quad \text{when } G \text{ is a group.}$$

$F[x, y, z]$ = polynomial algebra in x, y, z with coefficients in F (infinite dimensional)
 as distinguished from $Fx + Fy + Fz = \langle x, y, z \rangle_F = \{a_x x + a_y y + a_z z : a_x, a_y, a_z \in F\}$
 which is a 3-dimensional vector space

If R is an algebra over F and $S \subseteq R$ (any subset) then
 the centralizer of S in R is

$$C_R(S) = \{z \in R : zs = sz \text{ for all } s \in S\} \subseteq R \text{ subalgebra}$$

(Also called the commutant of S in R).

$$C_R(R) = Z(R)$$

Schur's Lemma (late 19th century)

Let R be a semisimple algebra over \mathbb{C} , e.g. $R = M(n, \mathbb{C})$, or $\bigoplus_{i=1}^k M(n_i, \mathbb{C})$, or $\mathbb{C}G$, $1 \leq k < \infty$.

Let M, N be R -^{simple} modules and $\phi : M \rightarrow N$ a homomorphism i.e. $\phi(rm + sm') = r\phi(m) + s\phi(m')$

For all $r, s \in R$; $m, m' \in M$.

(i) If $M \not\cong N$ as R -modules then $\phi = 0$.

There are no nonzero homomorphisms between simple modules.

(ii) If $M \cong N$, say $M = N$, then $\phi = cI$ for some $c \in \mathbb{C}$.

Proof (i) If $\phi \neq 0$ then there exists $v_0 \in M$ such that $\phi(v_0) \neq 0$, so $\underline{R\phi(v_0)} \subseteq \phi(M) \subseteq N$
 is a nonzero submodule of N . Since N is simple, $R\phi(v_0) = N$.

The kernel of $\phi : M \rightarrow N$ is a submodule of M .

Since M is simple $\ker \phi = 0$ or M . But $\phi \neq 0$ ($\phi(v_0) \neq 0$) we have
 $\ker \phi = 0$. By the first isomorphism theorem, $M \cong M / \underbrace{0}_{\ker \phi} \cong \phi(M) = N$

This contradicts $M \not\cong N$.

(ii) Let $\phi: M \rightarrow M$ be a homomorphism of the simple R -module M .
 In particular ϕ is a \mathbb{C} -linear transformation of a finite dimensional complex vector space so there exists $v_0 \in M, v_0 \neq 0$ such that $\phi(v_0) = cv_0$ for some $c \in \mathbb{C}$ (\mathbb{C} is alg. closed). Let $\tilde{\phi} = \phi - cI$ so $\tilde{\phi}$ is a homomorphism of R -algebras:

$$\tilde{\phi}(rm) = \phi(rm) - cirm = r\phi(m) - crm = r(\phi(m) - cm) = r\tilde{\phi}(m)$$

$$\tilde{\phi}(m+m') = \tilde{\phi}(m) + \tilde{\phi}(m') \quad (r \in R; m, m' \in M)$$

$$v_0 \in \ker \tilde{\phi} \neq 0 \Rightarrow \ker \tilde{\phi} = M \Rightarrow \tilde{\phi} = 0 \Rightarrow \phi = cI. \quad \square$$

Remark If $M \cong N$ as R -modules there is an isomorphism $A: M \rightarrow N$
 (A invertible $n \times n$ matrix, $n = \dim M = \dim N$)
 $A(rm) = rAm$ for all $r \in R$.

then the R -module homomorphisms $M \rightarrow N$ all have the form $cA, c \in \mathbb{C}$.

The choice of field \mathbb{C} is important for Schur's lemma e.g.

$$G = \{1, g, g^2, g^3\} \text{ cyclic of order 4, } \pi: G \rightarrow GL_2(\mathbb{R})$$

$$g \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$M = \mathbb{R}^2$ is an RG -module using π

$$\pi(g) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

π makes M into an R -module for $R = RG$

M is a simple module but the R -homomorphisms $M \rightarrow M$ are more than just $aI = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$; we have $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ ($a, b \in \mathbb{R}$)

Recall Schur's lemma:

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is an irreducible representation then the only $n \times n$ matrices commuting with $\pi(g)$, $g \in G$ are the scalar matrices λI_n ($\lambda \in \mathbb{C}$)

If π, π' are representations of, $\pi: G \rightarrow GL_m(\mathbb{C})$, $\pi': G \rightarrow GL_n(\mathbb{C})$ then π defines an action of G on \mathbb{C}^m . $g \in G$ acts on $v \in \mathbb{C}^m$ ($m \times 1$ column vector) as $gv = \pi(g)v \in \mathbb{C}^m$

and π' defines an action of G on \mathbb{C}^n , $gv = \pi'(g)v \in \mathbb{C}^n$

then a $\mathbb{C}G$ -homomorphism of the corresponding modules is (concretely) an $n \times m$ matrix $A \in \mathbb{C}^{n \times m}: \mathbb{C}^m \rightarrow \mathbb{C}^n$ such that

$$\begin{array}{ccc} \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \\ \pi(g) \downarrow & & \downarrow \pi'(g) \\ \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \end{array} \text{ commutes for every } g \in G \text{ i.e. } \pi'(g)A = A\pi(g)$$

for all $g \in G$

Such a matrix $A \in \mathbb{C}^{n \times m}$ is an intertwining operator (a linear transformation which respects/preserves the action of G).

i.e. $\alpha A = A \alpha$ for all $\alpha \in \mathbb{C}G$.

$\alpha(Av) = A(\alpha v)$ A is a homomorphism of $\mathbb{C}G$ -modules.

If $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$ are irreducible (simple) $\mathbb{C}G$ -modules (π, π' irred. representations):

- (i) inequivalent $\Rightarrow A = 0$
- (ii) $\pi \cong \pi'$ equivalent $\Rightarrow A = \lambda \cdot$ fixed invertible $n \times m$ matrix.

R semisimple algebra over \mathbb{C}

$\text{End}_R(R) = \{ \underbrace{R\text{-endomorphisms of } R}_{\text{into itself}} \}$ is the set of maps $\phi: R \rightarrow R$ such that $\phi(rx) = r\phi(x)$ for all $r, x \in R$

Lemma $\text{End}_R(R)$ is anti-isomorphic to R .

For every $a \in R$ we have $\phi_a: R \rightarrow R$, $\phi_a(x) = xa$.

$$\phi_a(rx) = rxa = r\phi_a(x) \Rightarrow \phi_a \in \text{End}_R(R).$$

The map $R \rightarrow \text{End}_R(R)$, $a \mapsto \phi_a \in \text{End}_R(R)$ is bijective.

If $a \in R$ such that $\phi_a = 0$ then $\phi_a(1) = 0$ so $a = 0$. So $a \mapsto \phi_a$ is one-to-one.

$$\phi_{ab}(r) = rab = \phi_b(ra) = \phi_b(\phi_a(r)) = \phi_b \circ \phi_a(r) \Rightarrow \phi_b \circ \phi_a = \phi_{ab}$$

If G is a group then $a \mapsto a^{-1}$ is an anti-automorphism $f(ab) = f(b)f(a)$
Similarly with an algebra.

If R is an algebra then the opposite algebra R° is the algebra with the same elements as R with same vector space structure; only the multiplication is replaced by $x * y = \underbrace{yx}_{\text{in } R}$ $x(yz) = x * (y * z) = (zy)x = z(yx) = (x * y) * z$

R° is not necessarily isomorphic to R . It is anti-isomorphic.

$G^\circ \cong G$ because we can compose two anti-isomorphisms to get an isomorphism
 $G \xrightarrow{\text{inverse}} G \rightarrow G^\circ$ If R is a division algebra then $R^\circ \cong R$ in the same way.

What are the R -endomorphisms of $R = \mathbb{C}G$? $\text{End}_R(R) = R^o$.

$R = I_1 \oplus I_2 \oplus \dots \oplus I_m$ as a direct sum of minimal left ideals.

If $\phi \in \text{End}_R(R)$ then we can represent ϕ using an $m \times m$ matrix over R :

$$\phi(r) =$$

$$r = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} \quad r_i \in I_i$$

$r = r_1 + \dots + r_m$ unique choice of $r_i \in I_i$

$\phi(r) = \phi(r_1) + \dots + \phi(r_m)$, decompose each term with respect to our decomposition

$$= \sum_{i=1}^m \left(\sum_{j=1}^m \phi_{ij} \cdot (r_j) \right)$$

$$\phi_{ij} \in \text{Hom}_R(I_i, I_j) = \begin{cases} 0 & \text{if } I_i \not\cong I_j \\ \mathbb{C}I_{n_i} & \text{if } I_i \cong I_j \end{cases}$$

Schur's Lemma

ϕ is represented by $m \times m$ matrix over \mathbb{C}

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \dots & \phi_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{m1} & \dots & \dots & \phi_{mm} \end{bmatrix} = \begin{bmatrix} \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} & & & \\ & \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} & & \\ & & \dots & \\ & & & \begin{matrix} * & \dots & * \\ * & \dots & * \\ * & \dots & * \end{matrix} \\ 0 & & & \\ & & & 0 \end{bmatrix} \in M(n_1, \mathbb{C}) \quad \begin{matrix} \uparrow \\ M(n_2, \mathbb{C}) \end{matrix}$$

$\text{End}_R(R)$ anti-iso. to $\bigoplus_{i=1}^k M(n_i, \mathbb{C})$

\parallel
 $R^o = R$ anti-iso

$R = \mathbb{C}G$ has an anti-isomorphism $\sum a_g g \mapsto \sum a_g g^{-1}$

$$R \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C})$$

p. 29 The center of the group algebra

G finite group $n = |G| = \dim \mathbb{C}G$

$Z(\mathbb{C}G) = \{ \alpha \in \mathbb{C}G : \alpha \gamma = \gamma \alpha \text{ for all } \gamma \in \mathbb{C}G \} \subseteq \mathbb{C}G$ subalgebra

$\dim Z(\mathbb{C}G) = k =$ number of conjugacy classes of G

K_1, K_2, \dots, K_k : conjugacy classes of $G = K_1 \sqcup \dots \sqcup K_k$
partition k

$$K_1 = \{1\}$$
$$\sum_{i=1}^k |K_i| = n = |G|$$