

Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
 characters representations of H ——— characters on G representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \rightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
 To get a class function on G , start with the trivial extension

$$\hat{\chi}: G \rightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \rightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \quad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \text{Ind}_H^G \chi)$$

$$\text{If } u \in G \text{ then } \chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wuw^{-1})$$

$$\begin{aligned} w^{-1} &= u^{-1}x^{-1} \\ w &= xu \\ wu &= x \end{aligned}$$

So χ^G is a class function on G . $= \chi^G(g)$

Note: Let T be a set of right coset representatives for H in G .

So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$

$$|G| = |H||T| \quad (\text{Lagrange's Theorem}), \quad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi: H \rightarrow \mathbb{C}$ is a class function on H ; T is a right transversal for H in G (set of right coset representatives)

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$x = ht, \quad \begin{matrix} h \in H \\ t \in T \end{matrix}$$

$$T = \{t_1, \dots, t_l\}$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \dots \sqcup Ht_l = \bigsqcup_{t \in T} Ht$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$$htgt^{-1}h^{-1} \in H$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$$\iff tgt^{-1} \in H$$

$|H|$ terms equal to $\hat{\chi}(tgt^{-1})$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^l \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H , $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G=H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \mapsto Ht_j = Ht_i g$.

$\chi^G = \left(\frac{1}{\#} \right)^G$ is the perm.

$$G \longrightarrow S_l$$

$$l = [G:H] = |T|.$$

character of G acting on cosets of H .

Ex. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

H:

| | | | |
|----------|-----|------|-------|
| $K_H(g)$ | 6 | 2 | 3 |
| g | (1) | (12) | (123) |
| χ_1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 |
| χ_3 | 2 | 0 | -1 |

check: $[\chi_3, \chi_3] = 1$

$G = S_4$:

| | | | | | |
|------------|-----|------|----------|-------|--------|
| $(C_G(g))$ | 24 | 4 | 8 | 3 | 4 |
| g | (1) | (12) | (12)(34) | (123) | (1234) |
| ψ_1 | 1 | 1 | 1 | 1 | 1 |
| ψ_2 | 1 | -1 | 1 | 1 | -1 |
| ψ_3 | 2 | 0 | 2 | -1 | 0 |
| ψ_4 | 3 | | | | |
| ψ_5 | 3 | | | | |
| ψ | 3 | 1 | 3 | 0 | 1 |

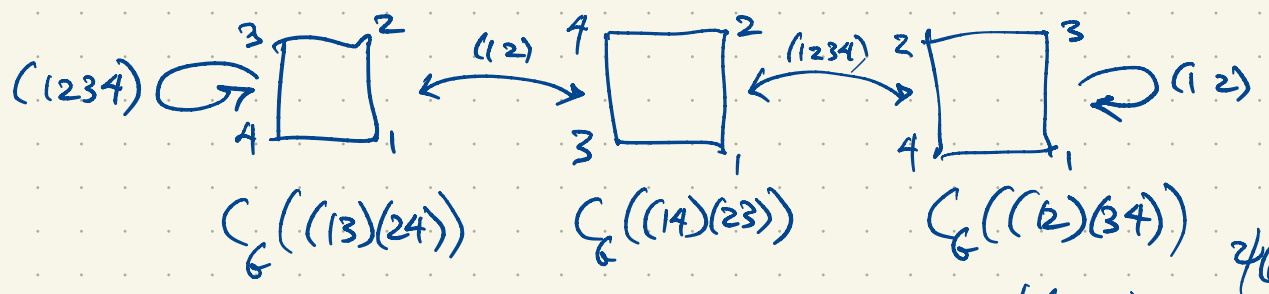
$[\psi_1, \psi_2] = 0$
 $\psi(1) = \psi_1(1) + \psi_2(1) = 1 + 1 = 2$
 $\psi = \sum_{i=1}^5 a_i \psi_i$
 $[\psi, \psi]_G = \frac{1^3}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$
 $= \frac{3 + 2 + 9 + 0 + 2}{8} = \frac{16}{8} = 2$
 $= \sum_{i=1}^5 a_i^2 = 1 + 1 + 0 + 0 + 0 = 2$
 $\psi = \psi_1 + \psi_3$
 (degree 1) (degree 2)
 $a_i = [\psi, \psi_i] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$
 $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2^2$

S_4 has $k=5$ conjugacy classes
 ... irreducible representations/characters
 of degree $n_1, n_2, \dots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2^2$

S_4 has normal subgroups

$1, S_4, A_4, K = \langle (12)(34), (13)(24) \rangle = \{ (1), (12)(34), (13)(24), (14)(23) \}$

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
 there is a permutation action $S_4 \rightarrow \text{Sym} \{ (12)(34), (13)(24), (14)(23) \} \cong S_3$
 with kernel K of order 4.



This gives a permutation representation of S_4 of degree 3.
 Its character is
 $\psi((1)) = 3$
 $\psi(k) = 3$ for $k \in K$
 $\psi((12)) = 1$
 $\psi((123)) = 0$
 $\psi((12)(34)) = 3$

$H = S_3$:

| | | | |
|----------|-----|------|-------|
| $K_H(g)$ | 6 | 2 | 3 |
| g | (1) | (12) | (123) |
| χ_1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 |
| χ_3 | 2 | 0 | -1 |

$G = S_4$:

| | | | | | |
|----------|-----|------|----------|-------|--------|
| (G) | 24 | 4 | 8 | 3 | 4 |
| g | (1) | (12) | (12)(34) | (123) | (1234) |
| ψ_1 | 1 | 1 | 1 | 1 | 1 |
| ψ_2 | 1 | -1 | 1 | 1 | -1 |
| ψ_3 | 2 | 0 | 2 | -1 | 0 |
| ψ_4 | 3 | -1 | -1 | 0 | 1 |
| ψ_5 | 3 | 1 | -1 | 0 | -1 |
| χ^G | 4 | -2 | 0 | 1 | 0 |

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity $\neq 1, 0, 0, 0$

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{ (1), (1234), (13)(24), (1432) \}$
right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi}: G \rightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G = \psi_2 + \psi_4$$

$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+1+0+2}{8} = 1$$

$$\chi^G(1) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \hat{\chi}((34)) + \hat{\chi}((14)) = -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) = 0 + 0 + 0 + 0 = 0$$

$$+ \hat{\chi}((23)(41))$$

$$+ \hat{\chi}((34)(12))$$

$$+ \hat{\chi}((41)(23))$$

$$\chi^G((123)) = \hat{\chi}((123)) + \hat{\chi}((234)) + \hat{\chi}((341)) + \hat{\chi}((412)) = 1 + 0 + 0 + 0 = 1$$

Frobenius Reciprocity let χ be a class function on $H \leq G$ and let ψ be a class function on G . Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

\uparrow $\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Proof $[\chi^G, \psi]_G = \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)}$

$x \in G = HT$
 $x = ht, \quad h \in H, t \in T$
 $|G| = |H||T|$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \frac{1}{|H|} \hat{\chi}(xgx^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \overline{\psi(g)} \quad \leftarrow htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(htgt^{-1}h^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$|H|$ identical terms for $h \in H$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{g \in G} \hat{\chi}(tgt^{-1}) \overline{\psi(g)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{u \in G} \hat{\chi}(u) \overline{\psi(t^{-1}ut)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(t^{-1}ht)} = \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

For each $t \in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$= \frac{1}{|G|} |T| \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= [\chi, \psi_H]_H \quad \square$$

If ψ is a class function on G then so is $\bar{\psi}$ where $\bar{\psi}(g) = \overline{\psi(g)} = \psi(g^{-1})$.
 Indeed if ψ is a character, $\psi(g) = \text{tr } \pi(g)$, $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\pi(g) \sim$ similar to $\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ in the case of finite groups) homomorphism

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity where $m = \text{exponent of } G = \text{lcm}(|g| : g \in G)$
 $\bar{\lambda}_i = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\bar{\lambda}_i \lambda_i = |\lambda_i|^2 = 1$
 $|\lambda_i|^m = 1$
 $|\lambda_i| = 1$

$$\psi(g) = \sum \lambda_i$$

$$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\psi(g)}$$

For every finite group G , the irreducible representations of G can be chosen to be unitary.

$$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}, \quad A^* = \bar{A}^T = \overline{A^T}$$

eg. for S_3 , we have an irreducible representation of degree 2.

| | |
|--|-----------|
| $\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$ | $\chi(g)$ |
| $(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | 2 |
| $(12) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ | 0 |
| $(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ | -1 |
| $(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ | -1 |
| $(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | 0 |
| $(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$ | 0 |

This representation uses integer matrix entries.

left-to-right composition

$$(13) = (123)(12)(132)$$

Character values:

$$\chi(g) = \text{tr } \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for \mathbb{C}^2 yields an equivalent representation using unitary matrices:

| | |
|--|-----------|
| | $\chi(g)$ |
| $(1) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ | 2 |
| $(123) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$ | -1 |
| $(132) \mapsto \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix}$ | -1 |
| $(12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | 0 |
| $(13) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix}$ | 0 |
| $(23) \mapsto \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$ | 0 |

$\omega = e^{2\pi i/3}$
 $\bar{\omega} = \omega^2 = \omega^{-1}$
 root of $x^2 + x + 1$
 $\omega^2 + \omega + 1 = 0$
 $\bar{\omega} + \omega = \omega^2 + \omega = -1$

To prove the claim (that every finite group has its irred. reps. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on \mathbb{C}^n is equivalent (under change of basis) to standard inner product $B(x, y) = \sum x_i \bar{y}_i$.

Any inner product has the form $(x, y) \mapsto x M y^*$ where $x = (x_1, \dots, x_n)$
 $M^* = M$ $n \times n$ matrix Hermitian, $\det M \neq 0$.
 $y^* = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$

If we perform an invertible change of variables $x \mapsto xA$ (A invertible $n \times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)
 then $B(xA, yA) = (xA)M(yA)^* = x \underbrace{(AMA^*)}_{\text{congruent to } M} y^*$

For all nonsingular Hermitian M , we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on

\mathbb{C}^n are equivalent by change of basis.

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x, y)$ on \mathbb{C}^n such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

- ✓ $B(x, x) \geq 0$, equality iff $x = 0$;
- ✓ $B(x, y)$ linear in x , conjugate linear in y ; $B(y, x) = \overline{B(x, y)}$;
- ✓ $B(x\pi(g), y\pi(g)) = B(x, y)$.

To obtain the inner product $B(x, y)$:

$$B(x, y) = \sum_{g \in G} (x \pi(g)) (y \pi(g))^* = \sum_{g \in G} x \pi(g) \pi(g)^* y^* = x M y^*$$

$$M = \sum_{g \in G} \pi(g) \pi(g)^* \text{ is Hermitian.}$$

$$B(x, x) = \sum_{g \in G} (x \pi(g)) (x \pi(g))^* = \sum_{g \in G} \|x \pi(g)\|^2 \geq 0$$

$B(x, x) > 0$ unless $x = 0$.

$$B(x \pi(g), y \pi(g)) = \sum_{u \in G} \underbrace{(x \pi(g) \pi(u))}_{\pi(w)} \underbrace{(y \pi(g) \pi(u))^*}_{\pi(w)} = \sum_{w \in G} (x \pi(w)) (y \pi(w))^* = B(x, y)$$

$w = gu$

Suppose $H \leq G$; let T be a right transversal for H in G i.e. a set of right coset representatives. Every $g \in G$ can be uniquely factored as $g = ht$, $h \in H$, $t \in T$. (Lagrange's theorem) $|G| = |H||T|$

G permutes the right cosets of H by right-multiplication:

$$Ht \mapsto Htg = Ht' \text{ for some } t' \in T.$$

This gives a permutation representation of G acting on the right cosets of H . If $d = |T| = [G : H]$ then we have a homomorphism $\pi: G \rightarrow S_d \subset GL_d(\mathbb{C})$ with perm. character $\psi(g) = \text{tr } \pi(g) = \text{no. of fixed points of } \pi(g)$

Theorem $\psi = (1_H)^G$ i.e. the perm. character is the induced character obtained from $1_H =$ principal character, $1_H(h) = 1$ for all $h \in H$, induced up to G .

Proof $\hat{1}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$

$$(1_H)^G(g) = \sum_{t \in T} \hat{1}_H(tgt^{-1}) = \text{number of } tgt^{-1} \quad (t \in T) \text{ which are in } H$$

$$= |\{t \in T : \underbrace{tgt^{-1} \in H}\}|$$

$$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Htg = Ht$$

\iff the coset Ht is fixed under right-multiplication by $g \in G$.

$$= \psi(g).$$

□

From the character table of G , we can see what all normal subgroups of G are.

$G = S_4$

| $(C_G(g))$ | 24 | 4 | 8 | 3 | 4 |
|------------|-----|------|----------|-------|--------|
| g | (1) | (12) | (12)(34) | (123) | (1234) |
| χ_1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 | 1 | -1 |
| χ_3 | 2 | 0 | 2 | -1 | 0 |
| χ_4 | 3 | -1 | -1 | 0 | 1 |
| χ_5 | 3 | 1 | -1 | 0 | -1 |

$\chi_2 : G \rightarrow \mathbb{C}^\times$ is a homomorphism
 Since $\chi_2(g) = \text{tr } \pi_2(g)$ of degree $\chi_2(1) = 1$.

Its kernel is $\{g \in G : \chi_2(g) = 1\}$
 $= \{ \text{elements conjugate to } (1), (12)(34) \text{ or } (123) \}$
 $= A_4$.

Another normal subgroup: all elements $g \in G$
 s.t. $\chi_3(g) = \chi_3(1) = 2$

Theorem Let $\pi : G \rightarrow GL_n(\mathbb{C})$ be a representation with character $\chi(g) = \text{tr } \pi(g)$.
 (Its degree is $n = \chi(1)$.) The kernel of χ defined by
 $\ker \chi = \ker \pi = \{g \in G : \pi(g) = I\} = \{g \in G : \chi(g) = \chi(1) = n\}$
 is a normal subgroup of G .

Note: π is a homomorphism; but χ is not a homomorphism unless $n=1$.

Proof If $g \in G$ then $\pi(g)$ is similar to $\begin{bmatrix} \epsilon_1 & & 0 \\ & \epsilon_2 & \\ 0 & & \ddots \\ & & & \epsilon_n \end{bmatrix}$, where $\epsilon_i^n = 1$ for all $i=1, 2, \dots, n$.

$\begin{bmatrix} \lambda & & \\ 0 & \lambda & \\ & & \ddots \\ 0 & & & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \\ & \lambda^k & \\ & & \ddots \\ 0 & & & \lambda^k \end{bmatrix}$

$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ & & \ddots \\ 0 & & & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

$\pi(g)$ is a root of $\chi - 1$

m is the exponent of G i.e. $\pi(g)^m = I$ for every $g \in G$

$$\Rightarrow \chi(g) = \sum \pi(g) = \varepsilon_1 + \dots + \varepsilon_n = n \iff \varepsilon_1 = \dots = \varepsilon_n = 1 \iff \pi(g) = I$$

$$|\varepsilon_i| = 1 \Rightarrow \operatorname{Re} \varepsilon_i \leq 1; \text{ equality iff } \varepsilon_i = 1.$$


For $G = S_4$, there are four normal subgroups and they all arise as $\ker \chi$ for some χ .

$$\ker \chi_5 = \{ () \}$$

$$\ker \chi_3 = \{ (), (12)(34), (13)(24), (14)(23) \}$$

$$\ker \chi_2 = A_4$$

$$\ker \chi_1 = S_4.$$

Recall: Let $H \leq G$. Then H is normal in G ($H \trianglelefteq G$) iff H is a union of conjugacy classes, iff H is an intersection of kernels of irreducible characters.

In this way we "read off" all the normal subgroups of G from the char. table.

eg. $G = \{\pm 1\} \times \{\pm 1\} = \{ (1,1), (1,-1), (-1,1), (-1,-1) \}$ Klein

| Char table | $ C_G(g) $ | | | |
|------------|------------|------------|------------|------------|
| | \uparrow | \uparrow | \uparrow | \uparrow |
| g | $(1,1)$ | $(1,-1)$ | $(-1,1)$ | $(-1,-1)$ |
| χ_1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | 1 | -1 | -1 |
| χ_3 | 1 | -1 | 1 | -1 |
| χ_4 | 1 | -1 | -1 | 1 |

Application to Frobenius groups.

Suppose $G \leq S_n$ (G permutes $\{1, 2, \dots, n\}$)

G permutes any set X , $G \leq \text{Sym}(X) = \{\text{permutations of } X\}$

More generally,
 X : any set.
eg. $X = \{1, 2, \dots, n\}$

G is transitive if for all $x, y \in X$, there exists $g \in G$ mapping $x \mapsto y$.

The stabilizer of a point $x \in X$ is $G_x = \{g \in G : g(x) = x\}$.

Of course $G_x \leq G$.

The orbit of $x \in X$ is $x^G = \{g(x) : g \in G\} \subseteq X$.

$$|x^G| = [G : G_x] = \frac{|G|}{|G_x|} \quad \text{or } G(x) \quad (\text{orbit-stabilizer formula}).$$

$x^G = X$ iff G is transitive.

G is a Frobenius group if

(i) G is transitive

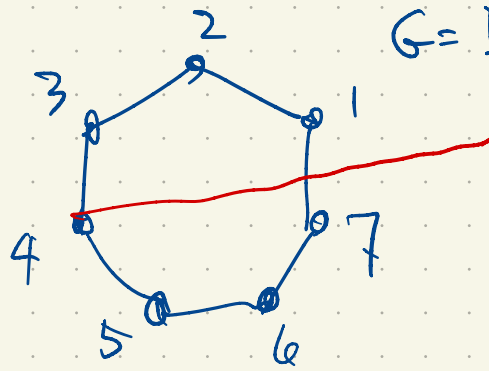
(ii) Point stabilizer is nontrivial

$$\left(|G_x| = \frac{|G|}{|X|} > 1 \right)$$

(iii) the stabilizer of any two points is trivial i.e. $G_x \cap G_y = 1$ for all $x \neq y$ in X .

If m is an odd positive integer then the symmetry group of a regular m -gon (i.e. the dihedral group of order $2m$) is a subgroup of S_m having m rotations, m reflections.

eg. $m=7$



$$G = D_7 = \langle (1234567), (17)(26)(35) \rangle$$

$$G_A = \langle (17)(26)(35) \rangle$$

$()$ = identity in G fixes 7 points;
 nontrivial rotations fix 0 "
 other elements fix 1 point.

This is a Frobenius group.

There are ^{exactly} two groups of order 21: the cyclic group, and a Frobenius group $\langle (1234567), (124)(365) \rangle < S_7$ transitive.

$$\downarrow \text{conjugate by } (124)(365)$$

$$(2461357) = (1357246) = (1234567)^2$$

Eg. $G =$ direct isometries of $\mathbb{R}^2 = \{ \text{translations} \} \vee \{ \text{rotations} \}$
 ↑ (orientation-preserving)

is a Frobenius group. there are lots of finite analogues of this example.

eg. $F = \mathbb{F}_{11} = \{\text{integers mod } 11\}$.

The affine general linear group on F^2 is a subgroup of S_{121} consisting of transformations $v \mapsto Av + b$, $A \in GL_2(F)$, $b \in F^2$.

$GL_2(F) = \{\text{invertible linear transformations on } F^2\}$

$$|GL_2(F)| = (11^2 - 1)(11^2 - 11) = 120 \cdot 110 = 13200$$

$$|AGL_2(F)| = 11^2 \cdot 13200 \quad \text{transitive on } F^2.$$

Stabilizer of $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in F^2$ is $GL_2(F)$ of order 13200.

This group $AGL_2(F)$ is not a Frobenius group e.g.

$$v \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{fixes all eleven vectors } \begin{bmatrix} a \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

The two distinct points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are fixed by more than just identity

in fact the subgroup $\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$ $b, c \in F, c \neq 0$ of order 110

Modify the example: $GL_2(F)$ has a subgroup isomorphic to $SL_2(\mathbb{F}_5)$ (order 120), ^{sharply} transitive on the 120 nonzero vectors

The ^{affine linear} transformations on F^2 of the form $v \mapsto Av + b$, $A \in$ subgp of $GL_2(\mathbb{F}_{11})$ isomorphic to $SL_2(\mathbb{F}_5)$, $b \in F^2$ forms a Frobenius group of order $121 \cdot 120$. Actually this example is sharply 2-transitive

On \mathbb{R}^2 , the transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ which are direct similarities is a Frob. gp., sharply 2-trans (invertible)

Let F be any field. An affine linear map $F \rightarrow F$ is a map $x \mapsto mx+b$, $m \neq 0$.

This group is $AGL_1(F)$.

Now if H is a subgroup of $F^\times = \{\text{nonzero elements of } F\}$ then the affine linear transformations $F \rightarrow F$, $x \mapsto mx+b$, $x \in H, b \in F$ is a Frobenius group. (sharply 2-trans. iff $H = F^\times$).

$$AGL_1(F) \cong \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} : m, b \in F, m \neq 0 \right\} < GL_2(F)$$

$$G = AGL_n(F) \cong \left\{ \left[\begin{array}{c|c} A & * \\ \hline 0 & 1 \end{array} \right] : A \in GL_n(F) \right\} < GL_{n+1}(F).$$

$$\text{has subgroups } K = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right] : A \in GL_n(F) \right\} \cong GL_n(F)$$

$$H = \left\{ \left[\begin{array}{c|c} I & * \\ \hline 0 & 1 \end{array} \right] \right\} \cong \text{translations of } F^n, \quad x \mapsto x+b$$

$$H \trianglelefteq AGL_n(F)$$

$$K = G_0 = \text{stabilizer of } 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

This is 2-transitive on F^n , not sharply in general.

If we replace $GL_n(F)$ by a subgroup having no nontrivial elements with eigenvalue 1 we get a Frobenius group.

Theorem (Frobenius) Every Frobenius group G has the form

$G = K \rtimes H$ i.e. subgroups H, K satisfy $G = KH$, $K \triangleleft G$, $K \cap H = 1$.

(Note: $KH = \{kh : h \in H, k \in K\}$ is a subgroup assuming one of them is normal.)

$$G/K = KH/K \cong H/(K \cap H) \cong H$$

$$H = G_0 \text{ where } 0 \in X \text{ any point in } X \\ = \{g \in G : g(0) = 0\}$$

What is K ? $K = \{1\} \cup \{\text{elements of } G \text{ which don't fix any point}\}$
 $= \{1\} \cup (G - \bigcup_{g \in G} (gHg^{-1})) \subseteq G$.

Since G is transitive, the stabilizer of any point $x \in X$ is conjugate to $H = G_0$. Why? Every point $x \in X$ has the form $x = g(0)$ for some $g \in G$.

If $h \in G_0 = H$ then $(ghg^{-1})(x) = (ghg^{-1})(g(0)) = gh(0) = \underbrace{g(0)}_{0 = g^{-1}(x)} = x$
 $h(0) = 0$

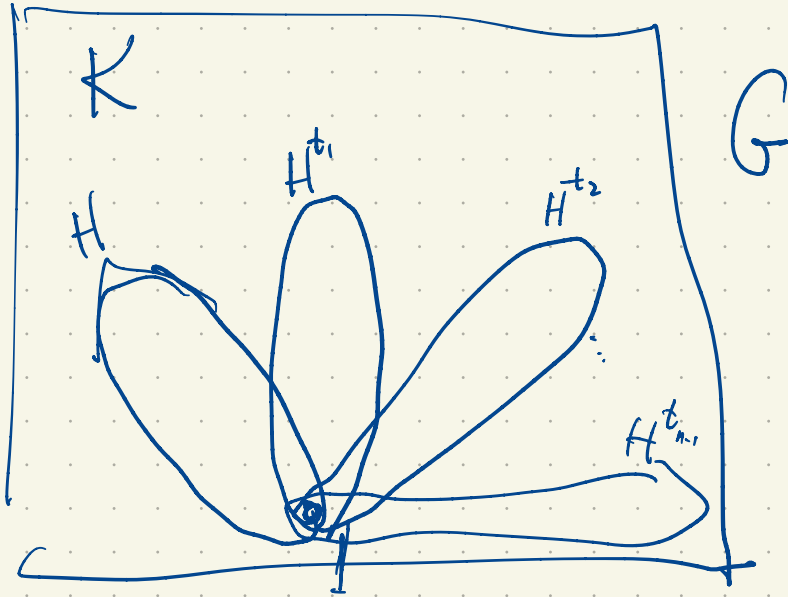
i.e. $ghg^{-1} \in G_x$. So $gHg^{-1} \subseteq G_x$ i.e. $gG_0g^{-1} \subseteq G_x$ $G_0 \subseteq g^{-1}G_xg$
 $G_0 \supseteq g^{-1}G_xg$

If $f \in G_x$ i.e. $f(x) = x$ then $(g^{-1}fg)(0) = g^{-1}f(x) = g^{-1}x = 0$

The problem is: the subset K defined above needs to be a subgroup.
If I can show this, normality follows immediately.

In S_5 (left-to-right),

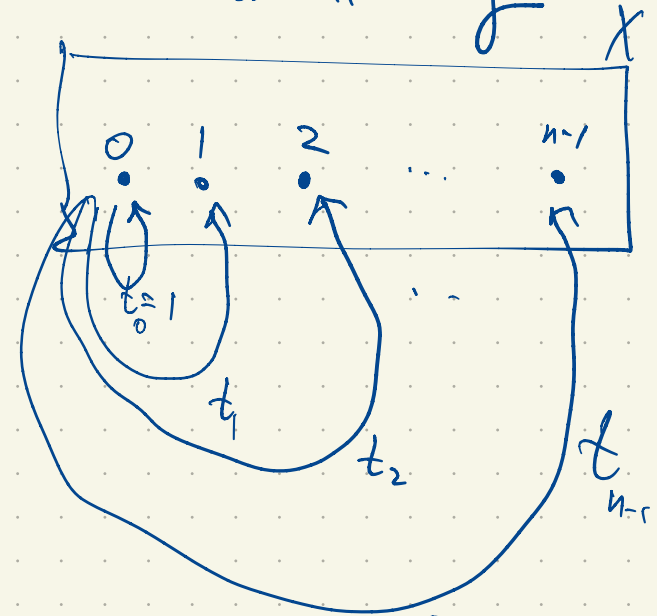
$$(12345)(15342) = (243)$$



$$H = G_0 = \{g \in G : g(0) = 0\}$$

$$0 \in X$$

G permutes X transitively



Any two stabilizers $t_i^{-1}Ht_i$, $t_j^{-1}Ht_j$ are disjoint for $i \neq j$:

$$H^{t_i} \cap H^{t_j} = 1 \text{ for } i \neq j$$

$$G_i = \{g \in G : g(i) = i\} = t_i^{-1}Ht_i$$

$$T = \{t_0, t_1, \dots, t_{n-1}\}$$

right transversal for H in G

Every conjugate of H has $|H|-1 = \frac{|G|}{n} - 1$ non-identity elements.

$$K = \{1\} \cup \left(G - \bigcup_{i=0}^{n-1} H^{t_i} \right)$$

$$G = \bigsqcup_{i=0}^{n-1} H^{t_i}$$

$$n|H| = |G|$$

$$|H| = \frac{|G|}{n}$$

$$|K| = 1 + \left(|G| - n \left(\frac{|G|}{n} - 1 \right) \right)$$

$$= 1 + |G| - 1 - |G| + n = n$$

Why is K a subgroup of G ? Normality is easy.
 We show that K is an intersection of kernels of characters of G .

Lemma Let θ be a class function on H . Then $\theta^G|_H = \theta$ i.e. $\theta^G(h) = \theta(h)$ for all $h \in H$.
 such that $\theta(1) = 0$.

Proof $\theta^G(g) = \sum_{i=0}^{n-1} \hat{\theta}(t_i^{-1}gt_i)$ where $\hat{\theta}(g) = \begin{cases} \theta(g), & \text{if } g \in H; \\ 0, & \text{if } g \notin H. \end{cases}$

For $g=1$, $\theta^G(1) = \sum_i \hat{\theta}(1) = \sum_i \theta(1) = 0 = \theta(1)$
 \uparrow
 $t_i^{-1}1t_i = 1$

For $h \in H$, $1 \neq h$, $\theta^G(h) = \sum_i \hat{\theta}(t_i^{-1}ht_i)$
 $= \hat{\theta}(h) = \theta(h)$
 as required. □

When is $t_i^{-1}ht_i \in H$?

i.e. $t_i^{-1}ht_i \in H \cap t_i^{-1}Ht_i$

Only if $i=0$, $t_0=1$, $t_0^{-1}ht_0 = h$

Now let ψ be any irreducible character of H , $\psi \neq \frac{1}{|H|} \mathbf{1}_H$ principal character $\frac{1}{|H|} \mathbf{1}_H(h) = 1$.

Define $\theta(h) = \psi(h) - \psi(1)$ i.e. $\theta = \psi - \psi(1)\mathbf{1}_H$.

So θ is a class function on H with $\theta(1) = \psi(1) - \psi(1) = 0$.

This yields an induced class function θ^G on G satisfying $\theta^G(h) = \theta(h) = \psi(h) - \psi(1)$ for all $h \in H$.

Define $\psi^* = \theta^G + \psi(1)\mathbf{1}_G$. This is a class function on G .

$\psi^*(g) = \theta^G(g) + \psi(1)$

We will prove that this is an irred. char. of G , and furthermore $\bigcap_{\psi \in \text{Irr}_H} \ker \psi^* = K$.

$$\psi \in \text{Irr}_H, \quad \psi \neq 1_H.$$

$$\theta = \psi - \psi(1)1_H$$

$$[\theta, \theta]_H = [\psi - \psi(1)1_H, \psi - \psi(1)1_H]_H$$

$$= [\psi, \psi]_H - [\psi, \psi(1)1_H]_H - [\psi(1)1_H, \psi]_H + [\psi(1)1_H, \psi(1)1_H]_H$$

$$= 1 - \underbrace{\psi(1)}_0 [\psi, 1_H]_H - \psi(1) \underbrace{[1_H, \psi]_H}_0 + \psi(1)^2 \underbrace{[1_H, 1_H]_H}_1$$

$$= 1 + \psi(1)^2$$

$$[\psi^*, \psi^*]_G = [\theta^G + \psi(1)1_G, \theta^G + \psi(1)1_G]_G =$$