



Math 5555

Abstract Algebra II

Book 3

For each $i \in \{1, 2, \dots, k\}$, we solve n_i equations (one for each $t \in \{1, \dots, n_i\}$)

in n_i unknowns $\lambda_{i,j}$, $j \in \{1, 2, \dots, n_i\}$.

In general the n_i symmetric polynomials $\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t$, $1 \leq t \leq n_i$, can be re-expressed in terms of the elementary symmetric polynomials $e_j = e_j(\lambda_{i,1}, \dots, \lambda_{i,n_i})$ which are the coefficients of

Use
(Newton's identities)

$$(x + \lambda_{i,1})(x + \lambda_{i,2}) \dots (x + \lambda_{i,n_i}) = x^{n_i} + e_1 x^{n_i-1} + e_2 x^{n_i-2} + \dots + e_{n_i} x + e_n$$

i.e. $e_n = \lambda_{i,1} \lambda_{i,2} \dots \lambda_{i,n_i}$

$$e_2 = \sum \lambda_{i,r} \lambda_{i,s}$$

$$e_1 = \lambda_{i,1} + \dots + \lambda_{i,n_i}$$

We will show: if $|G| = n$ then $\mathbb{C}G \cong \bigoplus_{i=1}^k M(n_i, \mathbb{C})$ (algebra isomorphism)

where $k =$ number of conjugacy classes in G .

The center of R (semisimple algebra) is $Z(R) = \{z \in R : zx = xz \text{ for all } x \in R\}$.

$M(n, \mathbb{C}) = \{n \times n \text{ complex matrices}\}$

$Z(R) \subseteq R$ is a subalgebra: a subspace which is also a subring.

$Z(M(n, \mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$

$$I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n \times n}$$

$$Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) = Z\left(\begin{bmatrix} * & & 0 \\ * & * & \\ 0 & & * \end{bmatrix}\right) = \left\{ \begin{bmatrix} \lambda_1 I_{n_1} & & 0 \\ & \lambda_2 I_{n_2} & \\ 0 & & \lambda_k I_{n_k} \end{bmatrix} : \lambda_1, \dots, \lambda_k \in \mathbb{C} \right\}$$

$$\dim \bigoplus_{i=1}^k M(n_i, \mathbb{C}) = \sum_{i=1}^k n_i^2 ; \quad \dim \left(Z\left(\bigoplus_{i=1}^k M(n_i, \mathbb{C})\right) \right) = k$$

$$\dim \mathbb{C}G = n = |G|$$

$$\dim \mathbb{Z}(CG) = k = \text{no. of conj. classes.}$$

Let $K_1, \dots, K_k \subset G$ be the conj. classes i.e. $G = K_1 \sqcup K_2 \sqcup \dots \sqcup K_k$

For $1 \leq i \leq k$, let $z_i = \sum_{g \in K_i} g = \text{sum of elements in } K_i$

$$z_i \in \mathbb{Z}(CG) \text{ because } \begin{aligned} gz_i &= z_i g \\ gz_i g^{-1} &= z_i \end{aligned}$$

$$\mathbb{Z}(CG) = \left\{ a_1 z_1 + \dots + a_k z_k : a_i \in \mathbb{C} \right\}$$

Given $z \in \mathbb{Z}(CG)$, say $z = \sum_{x \in G} a_x x \quad a_x \in \mathbb{C}$

For all $g \in G$, $\begin{aligned} gz &= zg \\ gzg^{-1} &= z \end{aligned}$

$$\mathbb{Z}G = \text{integral group ring of } G = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{Z} \right\}$$

$$\mathbb{Q}G = \text{rational group algebra} \quad \dots \quad \mathbb{Q}$$

$$\mathbb{R}G = \text{real group algebra} \quad \dots \quad \mathbb{R}$$

$$FG = \text{group algebra of } G \text{ over } F \quad \dots \quad F$$

$$F[G] = FG \quad \text{when } G \text{ is a group.}$$

$F[x, y, z]$ = polynomial algebra in x, y, z with coefficients in F (infinite dimensional)
 as distinguished from $Fx + Fy + Fz = \langle x, y, z \rangle_F = \{a_x x + a_y y + a_z z : a_x, a_y, a_z \in F\}$
 which is a 3-dimensional vector space

If R is an algebra over F and $S \subseteq R$ (any subset) then
 the centralizer of S in R is

$$C_R(S) = \{z \in R : zs = sz \text{ for all } s \in S\} \subseteq R \text{ subalgebra}$$

(Also called the commutant of S in R).

$$C_R(R) = Z(R)$$

Schur's Lemma (late 19th century)

Let R be a semisimple algebra over \mathbb{C} , e.g. $R = M(n, \mathbb{C})$, or $\bigoplus_{i=1}^k M(n_i, \mathbb{C})$, or $\mathbb{C}G$, $1/G < \infty$.

Let M, N be R -^{simple} modules and $\phi : M \rightarrow N$ a homomorphism i.e. $\phi(rm + sm') = r\phi(m) + s\phi(m')$

For all $r, s \in R$; $m, m' \in M$.

(i) If $M \not\cong N$ as R -modules then $\phi = 0$.

There are no nonzero homomorphisms between simple modules.

(ii) If $M \cong N$, say $M = N$, then $\phi = cI$ for some $c \in \mathbb{C}$.

Proof (i) If $\phi \neq 0$ then there exists $v_0 \in M$ such that $\phi(v_0) \neq 0$, so $\underline{R\phi(v_0)} \subseteq \phi(M) \subseteq N$
 is a nonzero submodule of N . Since N is simple, $R\phi(v_0) = N$.

The kernel of $\phi : M \rightarrow N$ is a submodule of M .

Since M is simple $\ker \phi = 0$ or M . But $\phi \neq 0$ ($\phi(v_0) \neq 0$) we have
 $\ker \phi = 0$. By the first isomorphism theorem, $M \cong M / \ker \phi \cong \phi(M) = N$

This contradicts $M \not\cong N$.

(ii) Let $\phi: M \rightarrow M$ be a homomorphism of the simple R -module M .
 In particular ϕ is a \mathbb{C} -linear transformation of a finite dimensional complex vector space so there exists $v_0 \in M, v_0 \neq 0$ such that $\phi(v_0) = cv_0$ for some $c \in \mathbb{C}$ (\mathbb{C} is alg. closed). Let $\tilde{\phi} = \phi - cI$ so $\tilde{\phi}$ is a homomorphism of R -algebras:

$$\tilde{\phi}(rm) = \phi(rm) - cirm = r\phi(m) - crm = r(\phi(m) - cm) = r\tilde{\phi}(m)$$

$$\tilde{\phi}(m+m') = \tilde{\phi}(m) + \tilde{\phi}(m') \quad (r \in R; m, m' \in M)$$

$$v_0 \in \ker \tilde{\phi} \neq 0 \Rightarrow \ker \tilde{\phi} = M \Rightarrow \tilde{\phi} = 0 \Rightarrow \phi = cI. \quad \square$$

Remark If $M \cong N$ as R -modules there is an isomorphism $A: M \rightarrow N$
 (A invertible $n \times n$ matrix, $n = \dim M = \dim N$)
 $A(rm) = rAm$ for all $r \in R$.

then the R -module homomorphisms $M \rightarrow N$ all have the form $cA, c \in \mathbb{C}$.

The choice of field \mathbb{C} is important for Schur's lemma e.g.

$$G = \{1, g, g^2, g^3\} \text{ cyclic of order 4, } \pi: G \rightarrow GL_2(\mathbb{R})$$

$$g \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$M = \mathbb{R}^2$ is an RG -module using π

$$\pi(g) \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

π makes M into an R -module for $R = RG$

M is a simple module but the R -homomorphisms $M \rightarrow M$ are more than just
 $aI = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$; we have $\begin{bmatrix} a & b \\ b & a \end{bmatrix} \quad (a, b \in \mathbb{R})$

Recall Schur's lemma:

If $\pi: G \rightarrow GL_n(\mathbb{C})$ is an irreducible representation then the only $n \times n$ matrices commuting with $\pi(g)$, $g \in G$ are the scalar matrices λI_n ($\lambda \in \mathbb{C}$)

If π, π' are representations of, $\pi: G \rightarrow GL_m(\mathbb{C})$, $\pi': G \rightarrow GL_n(\mathbb{C})$ then π defines an action of G on \mathbb{C}^m . $g \in G$ acts on $v \in \mathbb{C}^m$ ($m \times 1$ column vector) as $gv = \pi(g)v \in \mathbb{C}^m$

and π' defines an action of G on \mathbb{C}^n , $gv = \pi'(g)v \in \mathbb{C}^n$

then a $\mathbb{C}G$ -homomorphism of the corresponding modules is (concretely) an $n \times m$ matrix $A \in \mathbb{C}^{n \times m}$: $\mathbb{C}^m \rightarrow \mathbb{C}^n$ such that

$$\begin{array}{ccc} \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \\ \pi(g) \downarrow & & \downarrow \pi'(g) \\ \mathbb{C}^m & \xrightarrow{A} & \mathbb{C}^n \end{array} \text{ commutes for every } g \in G \text{ i.e. } \pi'(g)A = A\pi(g)$$

for all $g \in G$

Such a matrix $A \in \mathbb{C}^{n \times m}$ is an intertwining operator (a linear transformation which respects/preserves the action of G).

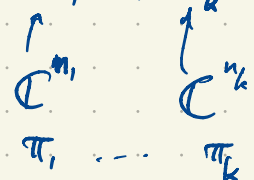
i.e. $\alpha A = A \alpha$ for all $\alpha \in \mathbb{C}G$.

$\alpha(Av) = A(\alpha v)$ A is a homomorphism of $\mathbb{C}G$ -modules.

If $V = \mathbb{C}^m$ and $W = \mathbb{C}^n$ are irreducible (simple) $\mathbb{C}G$ -modules (π, π' irred. representations):

- (i) inequivalent $\Rightarrow A = 0$
- (ii) $\pi \cong \pi'$ equivalent $\Rightarrow A = \lambda \cdot$ fixed invertible $n \times m$ matrix.

p.10 of handout. Theorem 4.4 (Wedderburn's Theorem)
 Let $R = \mathbb{C}G$, $|G| < \infty$. Let M_1, \dots, M_k be the distinct R -modules up to isomorphism.



The M_i -homogeneous part of R is the sum of all the left ideals of R isomorphic to M_i .

eg. $R = M(3, \mathbb{C})$. has only one simple module up to isomorphism, \mathbb{C}^3 .

$A \in R$ acts on $V = \mathbb{C}^3$ by $v \mapsto Av$.

$$R = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix} \quad \text{contains } \begin{bmatrix} a & b & 0 \\ b & b & 0 \\ c & c & 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}$$

Minimal Simple

Submodules (left ideals) of the regular representation are all isomorphic to \mathbb{C}^3 as R -modules.

The \mathbb{C}^3 -homogeneous part of this module is all of R .

$$R = \left[\begin{array}{|c|c|} \hline \boxed{1*} & \boxed{0} \\ \hline \boxed{**} & \boxed{**} \\ \hline \boxed{0} & \boxed{**} \\ \hline \end{array} \right] \text{ semisimple of dimension 4.}$$

$$\begin{aligned} \dim M_i &= n_i \\ \dim M_i(R) &= n_i^2 \end{aligned}$$

R has three iso. types of simple modules of dimension 1, 2, 3.

$$R = \begin{bmatrix} \boxed{1*} & \boxed{0} \\ \boxed{0} & \boxed{0} \end{bmatrix} \oplus \begin{bmatrix} \boxed{0} & \boxed{**} \\ \boxed{0} & \boxed{0} \end{bmatrix} \oplus \begin{bmatrix} \boxed{0} & \boxed{**} \\ \boxed{0} & \boxed{**} \end{bmatrix}$$

$M_1(R) \qquad M_2(R) \qquad M_3(R)$

If $n_i = \dim M_i$
 then the M_i -homo. part of $R = \mathbb{C}G$ is the sum of all submodules isomorphic to M_i .