Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
characters ... characters on G
representations of H ... representations of G.

Let χ be a class function on $H \le G$ i.e. $\chi: H \to \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.

To get a class function on G, start with the trivial extension

$$\hat{\chi}: G \longrightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \longrightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \operatorname{Ind}_H^G \chi\ )$$

$w^{-1} = u^{-1}x^{-1}$
$w = xu$
$wu = x$

If $u \in G$ then $\chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wgw^{-1})$

$= \chi^G(g)$.

So χ^G is a class function on G.

Note: Let T be a set of right coset representatives for H in G.
So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H$, $t \in T$

$|G| = |H||T|$ (Lagrange's Theorem), $|T| = \frac{|G|}{|H|} = [G : H]$.

$\chi : H \to \mathbb{C}$ is a class function on H; T is a right transversal for H in G (set of right coset representatives)

$T = \{t_1, \dots, t_\ell\}$

$G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_\ell$

$= \bigsqcup_{t \in T} Ht.$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$x = ht$, $h \in H$, $t \in T$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$

$$= \frac{1}{|H|} \sum_{t \in T} \underbrace{\sum_{h \in H} \hat{\chi}(tgt^{-1})}_{|H| \text{ terms equal to } \hat{\chi}(tgt^{-1})}$$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^{\ell} \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1$ = trivial (principal) character of H, $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G = H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \longmapsto Ht_i g = Ht_j$

$\chi^G = (1_H)^G$ is the perm. character of G acting on cosets of H. $\quad G \longrightarrow S_\ell \quad \ell = [G:H] = |T|$.

Eg. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \le G$.

H:

$\lvert C_H(g)\rvert$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

Check: $[\psi_3, \psi_3] = 1$

$[\psi_1, \psi_2] = 0$

$\underbrace{\psi(1)}_{3} = \underbrace{\psi_1(1)}_{1} + \underbrace{\psi_3(1)}_{2}$

$\psi = \sum_{i=1}^{5} a_i \psi_i$

$[\psi, \psi]_G = \frac{9}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$

$= \frac{3+2+9+0+2}{8} = \frac{16}{8}$

$= 2$

$= \sum_{i=1}^{5} a_i^2 = 1^2 + 1^2 + 0^2 + 0^2 + 0^2$

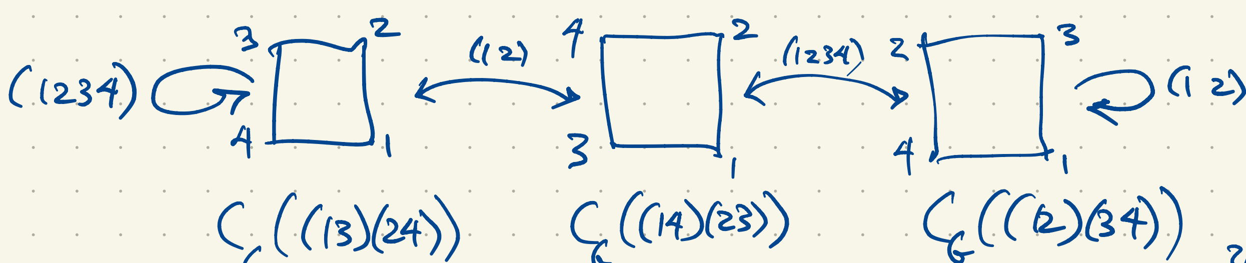
$\psi = \psi_1 + \psi_3$
(degree 1) (degree 2)

$a_1 = [\psi, \psi_1] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$

S_4 has $k = 5$ conjugacy classes
... irreducible representations/characters of degree $n_1, n_2, \dots, n_5 \ge 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G| = 24$

S_4 has normal subgroups
1, S_4, A_4, $K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$.

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
There is a permutation action $S_4 \longrightarrow \mathrm{Sym}\{(12)(34), (13)(24), (14)(23)\} \cong S_3$
with kernel K of order 4.



This gives a permutation representation of S_4 of degree 3.
Its permutation character is
$\psi(1) = 3$
$\psi(k) = 3$ for $k \in K$
$\psi((12)) = 1$
$\psi((1234)) = 1$ $\psi((123)) = 0$ $\psi((12)(34)) = 3$

$H=S_3$:

$\lvert C_H(g)\rvert$	6	2	3
g	$()$	(12)	(123)
χ_1	1	1	1
$\chi=\chi_2$	1	-1	1
χ_3	2	0	-1

$G=S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G,\chi^G]_G=\frac{16}{24}+\frac{4}{4}+\frac{0}{8}+\frac{1}{3}+\frac{0}{4}=2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity 1, 1, 0, 0, 0

$$[\chi^G,\psi_1]=\frac{4}{24}-\frac{2}{4}+\frac{0}{8}+\frac{1}{3}+\frac{0}{4}=0$$

$$[\chi^G,\psi_2]=\frac{4}{24}+\frac{2}{4}+\frac{0}{8}+\frac{1}{3}+\frac{0}{4}=1$$

$T=\langle(1234)\rangle$
$=\{(),(1234),(13)(24),(1432)\}$
right transversal for H in G

$\chi^G=\chi_2^G$ is a character on G

$$\chi^G(g)=\sum_{t\in T}\hat{\chi}(tgt^{-1})$$

where $\hat{\chi}:G\longrightarrow\mathbb{C}$

$$\hat{\chi}(g)=\begin{cases}\chi(g) & \text{if } g\in H\\ 0 & \text{if } g\notin H\end{cases}$$

$$\chi^G=\psi_2+\psi_4$$
$$\psi_4=\chi^G-\psi_2$$

$$[\psi_4,\psi_4]_G=\frac{9}{24}+\frac{1}{4}+\frac{1}{8}+\frac{0}{3}+\frac{1}{4}=\frac{3+2+1+0+2}{8}=1$$

$$\chi^G(1)=1+1+1+1=4$$

$$\chi^G((12))=\hat{\chi}((12))+\hat{\chi}((23))+\cancel{\hat{\chi}((34))}+\cancel{\hat{\chi}((14))}$$
$$=-1-1+0+0=-2$$

$$\chi^G((12)(34))=\hat{\chi}((12)(34))+\hat{\chi}((23)(41))+\hat{\chi}((34)(12))+\hat{\chi}((41)(23))=0+0+0+0=0$$

$$\chi^G((123))=\hat{\chi}((123))+\cancel{\hat{\chi}((234))}+\cancel{\hat{\chi}((341))}+\cancel{\hat{\chi}((412))}$$
$$=1+0+0+0=1$$

Frobenius Reciprocity Let χ be a class function on $H \leq G$ and let ψ be a class function on G. Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

$\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad \hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Proof

$$[\chi^G, \psi]_G = \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \frac{1}{|H|} \hat{\chi}(x^{-1}gx) \overline{\psi(g)}$$

$x \in G = HT$

$x = ht, \quad h \in H, \ t \in T$

$|G| = |H||T|$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \overline{\psi(g)}$$

$htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(htgt^{-1}h^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \underbrace{\hat{\chi}(tgt^{-1}) \overline{\psi(g)}}$$

$|H|$ identical terms ~~for~~ $h \in H$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{g \in G} \hat{\chi}(tgt^{-1}) \overline{\psi(g)} \right)$$

For each $t \in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{u \in G} \hat{\chi}(u) \overline{\psi(t^{-1}ut)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(t^{-1}ht)} = \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|G|} |T| \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi_H(h)}$$

$$= [\chi, \psi_H]_H. \qquad \square$$

If ψ is a class function on G then so is $\overline{\psi}$ where $\overline{\psi}(g) := \overline{\psi(g)} = \psi(g^{-1})$.

Indeed if ψ is a character, $\psi(g) = \operatorname{tr} \pi(g)$, $\pi : G \to GL_n(\mathbb{C})$ homomorphism

$\pi(g) \underset{\text{similar to}}{\sim} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ in the case of finite groups)

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity where m = exponent of G = $\operatorname{lcm}(|g| : g \in G)$

$\overline{\lambda_i} = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\quad \overline{\lambda_i}\lambda_i = |\lambda_i|^2 = 1$

$|\lambda_i|^m = 1$

$|\lambda_i| = 1$

$\psi(g) = \sum \lambda_i$

$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\psi(g)}$

For every finite group G, the irreducible representations of G can be chosen to be <u>unitary</u>.

$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}$, $\quad A^* = \overline{A}^T = \overline{A^T}$.

eg. for S_3, we have an irreducible representation of degree 2.

$\pi_3 : S_3 \to GL_2(\mathbb{C})$ $\qquad\qquad \chi(g)$

$() \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 2$

$(12) \longmapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \qquad 0$

$(123) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad -1$

$(132) \longmapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad -1$

$(13) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 0$

$(23) \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad 0$

This representation uses integer matrix entries.

left-to-right composition

$(13) = (123)(12)(132)$

Character values:

$$\chi(g) = \mathrm{tr}\, \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for $\mathbb{C}^2$ yields an equivalent representation using unitary matrices: $\qquad \chi(g)$

$() \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 2$

$(123) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}, \qquad -1$

$(132) \longmapsto \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} \qquad -1$

$(12) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 0$

$(13) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix}\begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix} \qquad 0$

$(23) \longmapsto \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix} \qquad 0$

$\omega = e^{2\pi i/3}$

$\bar{\omega} = \omega^2 = \omega^{-1}$

root of $x^2 + x + 1$

$\omega^2 + \omega + 1 = 0$

$\bar{\omega} + \omega = \omega^2 + \omega = -1$

To prove the claim (that every finite group has its irred. repns. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on $\mathbb{C}^n$ is equivalent (under change of basis) to standard inner product $B(x,y) = \sum x_i \overline{y_i}$.

Any inner product has the ~~form~~ $(x,y) \longmapsto xMy^*$ where $x=(x_1,\dots,x_n)$

$y^* = \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix}$

$M^* = M$ $n\times n$ matrix Hermitian, $\det M \neq 0$.

If we perform an invertible change of variables $x \longmapsto xA$ (A invertible $n\times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)

then $B(xA, yA) = (xA)M(yA)^* = x\underbrace{(AMA^*)}_{\text{congruent to } M}y^*$

For all nonsingular Hermitian M, we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on $\mathbb{C}^n$ are equivalent by change of basis.

Given a representation $\pi : G \longrightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x,y)$ on $\mathbb{C}^n$ such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

- $B(x,x) \geq 0$, equality iff $x=0$;
- $B(x,y)$ linear in x, conjugate linear in y; $B(y,x) = \overline{B(x,y)}$;
- $B(x\pi(g), y\pi(g)) = B(x,y)$.

To obtain the inner product $B(x,y)$:

$$B(x,y) = \sum_{g\in G} (x\pi(g))(y\pi(g))^* = \sum_{g\in G} x\pi(g)\pi(g)^* y^* = xMy^*$$

$$M = \sum_{g\in G} \pi(g)\pi(g)^* \quad \text{is Hermitian.}$$

$$B(x,x) = \sum_{g\in G}(x\pi(g))(x\pi(g))^* = \sum_{g\in G} \|x\pi(g)\|^2 \geq 0$$

$B(x,x) > 0$ unless $x = 0$.

$$B(x\pi(g), y\pi(g)) = \sum_{u\in G} (x\underbrace{\pi(g)\pi(u)}_{\pi(w)})(y\underbrace{\pi(g)\pi(u)}_{\pi(w)})^* = \sum_{w\in G} (x\pi(w))(y\pi(w))^* = B(x,y)$$

$w = gu$

Suppose $H \leq G$; let T be a right transversal for H in G i.e. a set of right coset representatives. Every $g\in G$ can be uniquely factored as $g = ht$, $h\in H$, $t\in T$. (Lagrange's theorem) $|G| = |H||T|$

G permutes the right cosets of H by right-multiplication:

$$Ht \longmapsto Htg = Ht' \text{ for some } t'\in T.$$

This gives a permutation representation of G acting on the right cosets of H. If $d = |T| = [G:H]$ then we have a homomorphism $\pi: G \longrightarrow S_d \subset GL_d(\mathbb{C})$. with perm. character $\psi(g) = \operatorname{tr}\pi(g) = $ no. of fixed points of $\pi(g)$

<u>Theorem</u> $\psi = (1_H)^G$ i.e. the perm. character is the induced character obtained from 1_H = principal character, $1_H(h) = 1$ for all $h \in H$, induced up to G.

<u>Proof</u> $$\hat{1}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$$

$(1_H)^G(g) = \sum_{t \in T} \hat{1}_H(tgt^{-1})$ = number of tgt^{-1} $(t \in T)$ which are in H

$= |\{t \in T : tgt^{-1} \in H\}|$

$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Hg = Ht$
$\iff$ the coset Ht is fixed under right-multiplication by $g \in G$.

$= \psi(g)$. $\square$.

From the character ~~table of~~ G, we can see what all normal subgroups of G are.

$G = S_4$:

$\lvert C_G(g)\rvert$	24	4	8	3	4
g	$()$	(12)	$(12)(34)$	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1

$\psi_2 : G \to \mathbb{C}^\times$ is a homomorphism since $\psi_2(g) = \operatorname{tr} \pi_2(g)$ of degree $\psi_2(1) = 1$.

Its kernel is $\{g \in G : \psi_2(g) = 1\}$
$= \{$elements conjugate to $()$, $(12)(34)$ or $(123)\}$
$= A_4$.

Another normal subgp: all elements $g \in G$ s.t. $\psi_3(g) = \psi_3(1) = 2$

__Theorem__ Let $\pi : G \to GL_n(\mathbb{C})$ be a representation with character $\chi(g) = \operatorname{tr} \pi(g)$. (Its degree is $n = \chi(1)$.) The kernel of χ defined by

$$\ker \chi = \ker \pi = \{g \in G : \pi(g) = I\} = \{g \in G : \chi(g) = \chi(1) = n\}$$

is a normal subgroup of G.

Note: π is a homomorphism; but χ is not a homomorphism unless $n = 1$.

__Proof__ If $g \in G$ then $\pi(g)$ is similar to $\begin{bmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \ddots \; \varepsilon_n \end{bmatrix}$, where $\boxed{\varepsilon_i^m} = 1$ for all $i = 1, 2, \ldots, n$.

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1}\alpha \\ 0 & \lambda^k \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

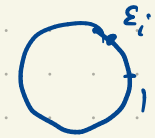
$\pi(g)$ is a root of $x^m - 1$

m is the exponent of G

$\pi(g) \in GL_n(\mathbb{C})$ has order dividing m. i.e. $g^m = 1$ for every $g \in G$

$\Longrightarrow \chi(g) = \operatorname{tr} \pi(g) = \varepsilon_1 + \cdots + \varepsilon_n = n \iff \varepsilon_1 = \cdots = \varepsilon_n = 1. \iff \pi(g) = I$

$|\varepsilon_i| = 1 \Longrightarrow \operatorname{Re} \varepsilon_i \le 1$; equality iff $\varepsilon_i = 1$.



For $G = S_4$, there are four normal subgroups and they all arise as $\ker \chi$ for some χ.

$\ker \chi_5 = \{ () \}$

$\ker \chi_3 = \{ (), (12)(34), (13)(24), (14)(23) \}$

$\ker \chi_2 = A_4$

$\ker \chi_1 = S_4$.

Recall: Let $H \le G$. Then H is normal in G $(H \trianglelefteq G)$ iff H is a union of conjugacy classes, iff H is an intersection of kernels of irreducible characters.

In this way we "read off" all the normal subgroups of G from the char. table.

eg. $G = \{\pm 1\} \times \{\pm 1\} = \{ (1,1), (1,-1), (-1,1), (-1,-1) \}$ Klein

Char table

$\lvert C_G(g) \rvert$	4	4	4	4
g	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Application to Frobenius groups.

Suppose $G \leq S_n$ (G permutes $\{1,2,\cdots,n\}$). More generally, G permutes any set X, $G \leq \mathrm{Sym}(X) = \{\text{permutations of } X\}$ X: any set. eg. $X = \{1,2,\cdots,n\}$

G is <u>transitive</u> if for all $x,y \in X$, there exists $g \in G$ mapping $x \mapsto y$.

The <u>stabilizer</u> of a point $x \in X$ is $G_x = \{g \in G : g(x) = x\}$.

Of course $G_x \leq G$.

The <u>orbit</u> of $x \in X$ is $x^G = \{g(x) : g \in G\} \subseteq X$.
or $G(x)$

$$|x^G| = [G : G_x] = \frac{|G|}{|G_x|} \quad \text{(orbit-stabilizer formula)}.$$

$x^G = X$ iff G is transitive.

G is a <u>Frobenius group</u> if

(i) G is transitive

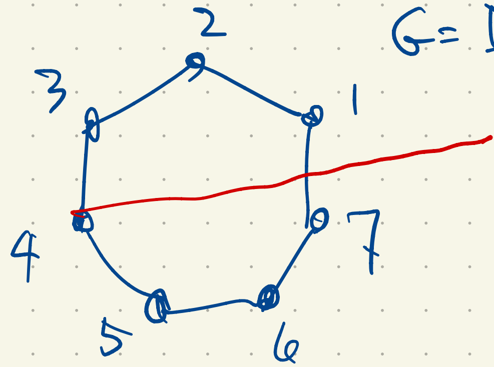
(ii) Point stabilizer is nontrivial $\left(|G_x| = \dfrac{|G|}{|X|} > 1 \right)$

(iii) The stabilizer of any two points is trivial ie. $G_x \cap G_y = 1$ for all $x \neq y$ in X.

If m is an odd positive integer then the symmetry group of a regular m-gon (i.e. the dihedral group of order $2m$) is a subgroup of S_m having

- m rotations,
- m reflections.

eg. $m=7$



$G = D_7 = \langle (1234567), (17)(26)(35) \rangle$

$G_4 = \langle (17)(26)(35) \rangle$

$()$ = identity in G fixes 7 points;
nontrivial rotations fix 0 " ;
other elements fix 1 point.

This is a Frobenius group.

There are exactly two groups of order 21: the cyclic group, and a Frobenius group $\langle (1234567), (124)(365) \rangle < S_7$ transitive.

$\downarrow$ conjugate by $(124)(365)$

$$(2461357) = (1357246) = (1234567)^2.$$

Eg. G = direct isometries of $\mathbb{R}^2$ = $\{\text{translations}\} \cup \{\text{rotations}\}$
(orientation-preserving)
is a Frobenius group. There are lots of finite analogues of this example.

e.g. $F = \mathbb{F}_{11}$ = {integers mod 11}.

The affine general linear group on F^2 is a subgroup of S_{121} consisting of transformations $v \longmapsto Av + b$, $A \in GL_2(F)$, $b \in F^2$.

$GL_2(F)$ = { invertible linear transformations on F^2 }

$|GL_2(F)| = (11^2-1)(11^2-11) = 120 \cdot 110 = 13200$

$|AGL_2(F)| = 11^2 \cdot 13200$ transitive on F^2.

Stabilizer of $0 = \begin{bmatrix}0\\0\end{bmatrix} \in F^2$ is $GL_2(F)$ of order 13200.

This group $AGL_2(F)$ is not a Frobenius group e.g.

$v \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix} v + \begin{bmatrix}0\\0\end{bmatrix}$ fixes all eleven vectors $\begin{bmatrix}a\\0\end{bmatrix} \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix}\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix}$

The two distinct points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ are fixed by more than just identity

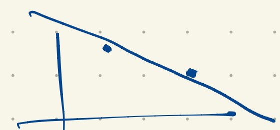
in fact the subgroup $\begin{bmatrix}1&b\\0&c\end{bmatrix}$ of order 110, $b, c \in F$, $c \neq 0$

Modify the example: $GL_2(F)$ has a subgroup isomorphic to $SL_2(\mathbb{F}_5)$ (order 120), ~~transitive~~ sharply transitive on the 120 nonzero vectors

The affine linear ~~transformations~~ on F^2 of the form $v \longmapsto Av+b$, $A \in$ subgp of $GL_2(\mathbb{F}_{11})$ isomorphic to $SL_2(\mathbb{F}_5)$, $b \in \mathbb{F}_{11}^2$ forms a Frobenius group of order $121 \cdot 120$. Actually this example is sharply 2-transitive

On $\mathbb{R}^2$, the transformations $\mathbb{R}^2 \to \mathbb{R}^2$ which are direct similarities is a Frob. gp., sharply 2-trans (invertible)

Let F be any field. An affine linear map $F \to F$ is a map $x \mapsto mx+b$, $m \neq 0$.



This group is $AGL_1(F)$.

Now if H is a subgroup of $F^\times = \{\text{nonzero elements of } F\}$ then the affine linear transformations $F \to F$, $x \mapsto mx+b$, $x \in H$, $b \in F$ is a Frobenius group. (sharply 2-trans. iff $H = F^\times$).

$$AGL_1(F) \cong \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} : m, b \in F, \ m \neq 0 \right\} < GL_2(F)$$

$$G = AGL_n(F) \cong \left\{ \left[\begin{array}{c|c} A & \begin{smallmatrix} * \\ * \\ * \end{smallmatrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] : A \in GL_n(F) \right\} < GL_{n+1}(F).$$

has subgroups
$$K = \left\{ \left[\begin{array}{c|c} A & \begin{smallmatrix} 0 \\ \vdots \\ 0 \end{smallmatrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] : A \in GL_n(F) \right\} \cong GL_n(F)$$

$$H = \left\{ \left[\begin{array}{c|c} I & \begin{smallmatrix} * \\ * \\ * \end{smallmatrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \right\} \cong \text{translations of } F^n, \quad x \mapsto x+b$$

$$H \trianglelefteq AGL_n(F)$$

$$K = G_0 = \text{stabilizer of } 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is 2-transitive on F^n, not sharply in general.

If we replace $GL_n(F)$ by a subgroup having no nontrivial elements with eigenvalue 1 we get a Frobenius group.

Theorem (Frobenius) Every Frobenius group G has the form

$G = K \rtimes H$ i.e. subgroups H, K satisfy $G = KH$, $K \trianglelefteq G$, $K \cap H = 1$.

(Note: $KH = \{kh : h \in H, k \in K\}$ is a subgroup assuming one of them is normal.)

$H = G_0$ where $0 \in X$ any point in X
$= \{g \in G : g(0) = 0\}$

$$G/K = {KH}/K \cong H/{K \cap H} \cong H.$$

what is K? $K = \{1\} \cup \{$elements of G which don't fix any point$\}$
$$= \{1\} \cup \left(G - \bigcup_{g \in G} (gHg^{-1})\right) \subseteq G.$$

Since G is transitive, the stabilizer of any point $x \in X$ is conjugate to $H = G_0$. Why? Every point $x \in X$ has the form $x = g(0)$ for some $g \in G$.

If $h \in G_0 = H$ then $(ghg^{-1})(x) = (ghg^{-1})(g(0)) = gh(0) = g(0) = x$
$h(0) = 0$ $\qquad 0 = g^{-1}(x)$

i.e. $ghg^{-1} \in G_x$. So $gHg^{-1} \subseteq G_x$ i.e. $gG_0g^{-1} \subseteq G_x$

$G_0 \subseteq g^{-1}G_x g$
$G_0 \supseteq g^{-1}G_x g$

If $f \in G_x$ i.e. $f(x) = x$ then $(g^{-1}fg)(0) = g^{-1}f(x) = g^{-1}x = 0$.

The problem is: the subset K defined above needs to be a subgp.
If I can show this, normality follows immediately.

In S_5 (left-to-right), $(1\,2\,3\,4\,5)(1\,5\,3\,4\,2) = (2\,4\,3)$