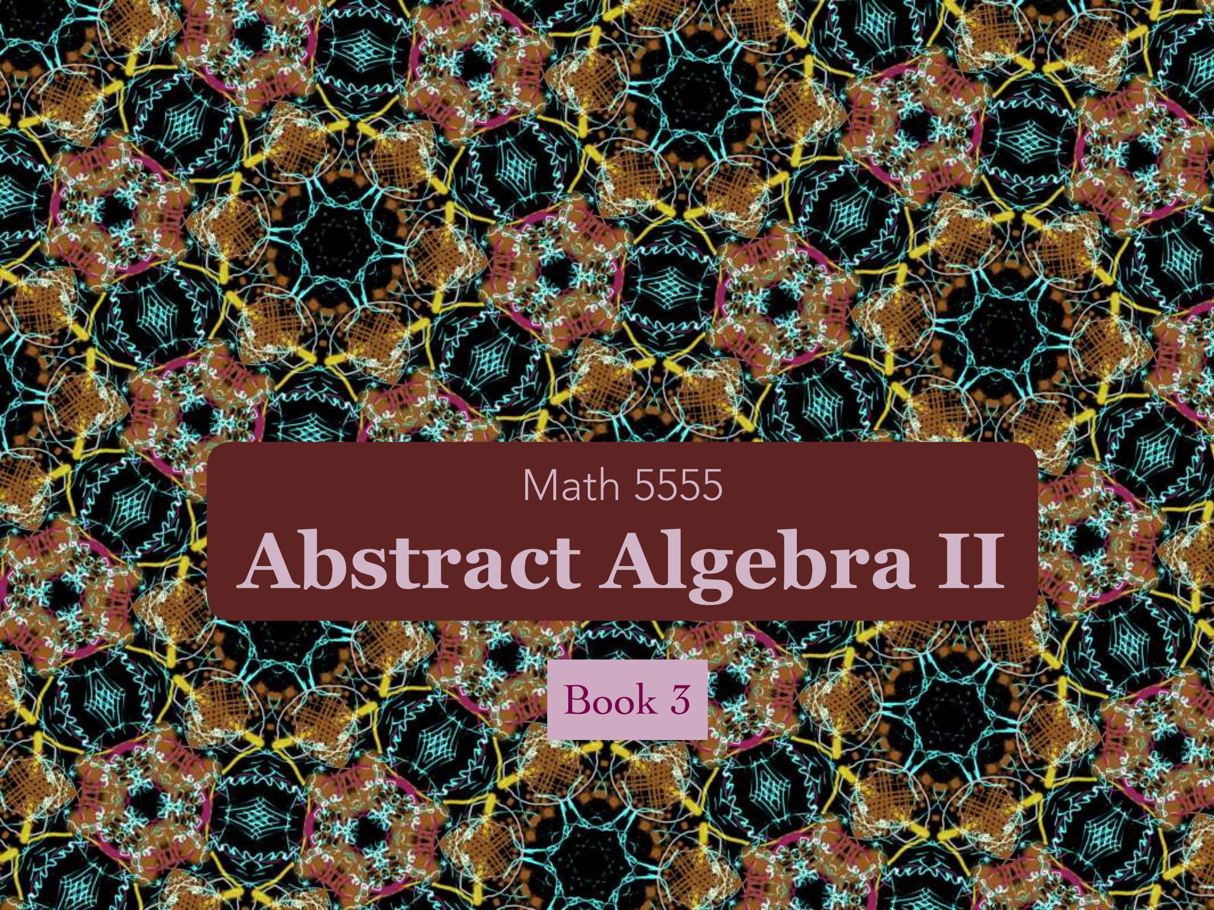

Math 5555

Abstract Algebra II

Book 3

For each $i \in \{1,2,\dots,k\}$, we solve n_i equations (one for each $t \in \{1,\dots,n_i\}$)

in n_i unknowns $\lambda_{i,j}$, $j \in \{1,2,\dots,n_i\}$.

In general the n_i symmetric polynomials $\lambda_{i,1}^t + \cdots + \lambda_{i,n_i}^t$ $\quad 1 \le t \le n_i$

can be re-expressed in terms of the elementary symmetric polynomials

Use (Newton's identies)

$e_j = e_j(\lambda_{i,1},\dots,\lambda_{i,n_i})$ which are the coefficients of

$$(x+\lambda_{i,1})(x+\lambda_{i,2})\cdots(x+\lambda_{i,n_i}) = x^{n_i} + e_1 x^{n_i-1} + e_2 x^{n_i-2} + \cdots + e_{n-1}x + e_n$$

i.e. $e_n = \lambda_{i,1}\lambda_{i,2}\cdots\lambda_{i,n_i}$

$\vdots$

$e_2 = \sum \lambda_{i,r}\lambda_{i,s}$

$e_1 = \lambda_{i,1} + \cdots + \lambda_{i,n_i}$

We will show: if $|G| = n$ then $\mathbb{C}G \cong \bigoplus_{i=1}^{k} M(n_i,\mathbb{C})$ (algebra isomorphism)

where k = number of conjugacy classes in G.

$M(n,\mathbb{C}) = \{n\times n \text{ complex matrices}\}$

The center of R (semisimple algebra) is

$Z(R) = \{z \in R : zx = xz \text{ for all } x \in R\}$.

$Z(M(n,\mathbb{C})) = \{\lambda I : \lambda \in \mathbb{C}\}$

$I = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}_{n\times n}$

$Z(R) \subseteq R$ is a subalgebra: a subspace which is also a subring.

$$Z\left(\bigoplus_{i=1}^{k} M(n_i,\mathbb{C})\right) = Z\left(\begin{bmatrix} \underset{n_1\times n_1}{\boxed{*}} & & & 0 \\ & \underset{n_2\times n_2}{\boxed{*}} & & \\ & & \ddots & \\ 0 & & & \underset{n_k\times n_k}{\boxed{*}} \end{bmatrix}\right) = \left\{ \begin{bmatrix} \boxed{\lambda_1 I_{n_1}} & & & 0 \\ & \boxed{\lambda_2 I_{n_2}} & & \\ & & \ddots & \\ 0 & & & \boxed{\lambda_k I_{n_k}} \end{bmatrix} : \lambda_1,\dots,\lambda_k \in \mathbb{C} \right\}$$

$\dim \bigoplus_{i=1}^{k} M(n_i,\mathbb{C}) = \sum_{i=1}^{k} n_i^2$; $\dim\left(Z\left(\bigoplus_{i=1}^{k} M(n_i,\mathbb{C})\right)\right) = k$

$\dim \mathbb{C}G = n = |G|$

$\dim Z(\mathbb{C}G) = k =$ no. of conj. classes.

Let $K_1, \cdots K_k \subset G$ be the conj. classes i.e. $G = \underset{\{1\}}{\underset{\parallel}{K_1}} \sqcup K_2 \sqcup \cdots \sqcup K_k$

For $1 \leq i \leq k$, let $z_i = \sum_{g \in K_i} g =$ sum of elements in K_i

$z_i \in Z(\mathbb{C}G)$ because $gz_i = z_i g$

$gz_i g^{-1} = z_i$

$Z(\mathbb{C}G) = \{ a_1 z_1 + \cdots + a_k z_k : a_i \in \mathbb{C} \}$

Given $z \in Z(\mathbb{C}G)$, say $z = \sum_{x \in G} a_x x$ $\quad a_x \in \mathbb{C}$

For all $g \in G$, $gz = zg$

$gzg^{-1} = z$

$\mathbb{Z}G$ = integral group ring of G $= \{ \sum_{g \in G} a_g g : a_g \in \mathbb{Z} \}$

$\mathbb{Q}G$ = rational group algebra " " - - - - $\mathbb{Q}$

$\mathbb{R}G$ = real group algebra - - - $\mathbb{R}$

FG = group algebra of G over F $\quad F$

$F[G] = FG$ when G is a group.

$F[x,y,z]$ = polynomial algebra in x,y,z with coefficients in F (infinite dimensional)
as distinguished from $Fx + Fy + Fz = \langle x,y,z\rangle_F = \{a_x x + a_y y + a_z z : a_x, a_y, a_z \in F\}$
which is a 3-dimensional vector space

If R is an algebra over F and $S \subseteq R$ (any subset) then
the centralizer of S in R is

$$C_R(S) = \{z \in R : zs = sz \text{ for all } s \in S\} \subseteq R \text{ subalgebra}$$

(Also called the commutant of S in R). $\qquad C_R(R) = Z(R)$

Schur's lemma (late 19th century)
Let R be a simple algebra over $\mathbb{C}$ of finite dimension.
vector space
ring with no 2-sided ideals other than 0 and R