



Math 5555

Abstract Algebra II

Book 2

Induction takes class functions on H to class functions on G
 characters representations of H ——— characters on G representations of G

Let χ be a class function on $H \leq G$ i.e. $\chi: H \rightarrow \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
 To get a class function on G , start with the trivial extension

$$\hat{\chi}: G \rightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G: G \rightarrow \mathbb{C}$:

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \quad (\chi^G \text{ is induced from } \chi, \quad \chi^G = \text{Ind}_H^G \chi)$$

$$\text{If } u \in G \text{ then } \chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wuw^{-1})$$

$$\begin{aligned} w^{-1} &= u^{-1}x^{-1} \\ w &= xu \\ wu &= x \end{aligned}$$

So χ^G is a class function on G . $= \chi^G(g)$

Note: Let T be a set of right coset representatives for H in G .

So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H, t \in T$

$$|G| = |H||T| \quad (\text{Lagrange's Theorem}), \quad |T| = \frac{|G|}{|H|} = [G:H].$$

$\chi: H \rightarrow \mathbb{C}$ is a class function on H ; T is a right transversal for H in G (set of right coset representatives)

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$x = ht, \quad \begin{matrix} h \in H \\ t \in T \end{matrix}$$

$$T = \{t_1, \dots, t_l\}$$

$$G = Ht_1 \sqcup Ht_2 \sqcup \dots \sqcup Ht_l$$

$$= \bigsqcup_{t \in T} Ht$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$$htgt^{-1}h^{-1} \in H$$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(tgt^{-1})$$

$$\iff tgt^{-1} \in H$$

$|H|$ terms equal to $\hat{\chi}(tgt^{-1})$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^l \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of H , $\chi(h) = 1$.

χ^G isn't the principal character of G unless $G=H$.

G permutes the right cosets of H by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \mapsto Ht_j = Ht_i g$.

$\chi^G = \left(\underset{\#}{1} \right)_G$ is the perm.

$$G \longrightarrow S_l$$

$$l = [G:H] = |T|.$$

character of G acting on cosets of H .

Ex. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \leq G$.

$S_3 = H \leq G$

H:

$K_H(g)$	6	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

check: $[\chi_3, \chi_3] = 1$

$G = S_4$:

$(C_G(g))$	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3				
ψ_5	3				
ψ	3	1	3	0	1

$[\psi_1, \psi_2] = 0$

$\psi(1) = \psi_1(1) + \psi_2(1)$

$\psi = \sum_{i=1}^5 a_i \psi_i$

$[\psi, \psi]_G = \frac{1^3}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$

$= \frac{3 + 2 + 9 + 0 + 2}{8} = \frac{16}{8} = 2$

$= \sum_{i=1}^5 a_i^2 = 1 + 1 + 0 + 0 + 0 = 2$

$\psi = \psi_1 + \psi_3$

(degree 1) (degree 2)

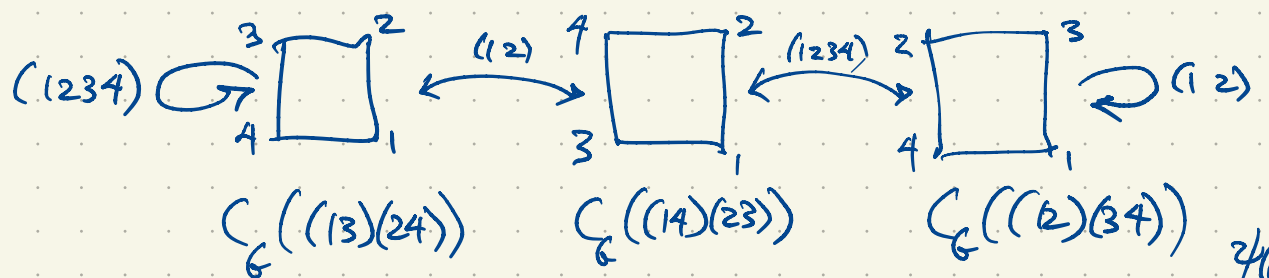
$a_i = [\psi, \psi_i] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$

S_4 has $k=5$ conjugacy classes
 ... irreducible representations/characters
 of degree $n_1, n_2, \dots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = 16 = 2^2$

S_4 has normal subgroups

$1, S_4, A_4, K = \langle (12)(34), (13)(24) \rangle = \{ (1), (12)(34), (13)(24), (14)(23) \}$

S_4 permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
 there is a permutation action $S_4 \rightarrow \text{Sym} \{ (12)(34), (13)(24), (14)(23) \} \cong S_3$
 with kernel K of order 4.



This gives a permutation representation of S_4 of degree 3.
 Its character is

$\psi((1)) = 3$
 $\psi(k) = 3$ for $k \in K$
 $\psi((12)) = 1$
 $\psi((123)) = 0$
 $\psi((12)(34)) = 3$

$H = S_3$:

$K_H(g)$	6	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$G = S_4$:

(G)	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
ψ_1	1	1	1	1	1
ψ_2	1	-1	1	1	-1
ψ_3	2	0	2	-1	0
ψ_4	3	-1	-1	0	1
ψ_5	3	1	-1	0	-1
χ^G	4	-2	0	1	0

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow \chi^G$ has irreducible constituents of multiplicity $\neq 1, 0, 0, 0$

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{ (1), (1234), (13)(24), (1432) \}$
right transversal for H in G

$\chi^G = \chi_2^G$ is a character on G

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi}: G \rightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G = \psi_2 + \psi_4$$

$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3+2+1+0+2}{8} = 1$$

$$\chi^G(1) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \hat{\chi}((34)) + \hat{\chi}((14)) = -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) = 0 + 0 + 0 + 0 = 0$$

$$+ \hat{\chi}((23)(41))$$

$$+ \hat{\chi}((34)(12))$$

$$+ \hat{\chi}((41)(23))$$

$$\chi^G((123)) = \hat{\chi}((123)) + \hat{\chi}((234)) + \hat{\chi}((341)) + \hat{\chi}((412)) = 1 + 0 + 0 + 0 = 1$$

Frobenius Reciprocity let χ be a class function on $H \leq G$ and let ψ be a class function on G . Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

\uparrow $\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

Proof $[\chi^G, \psi]_G = \frac{1}{|G|} \sum_{g \in G} \chi^G(g) \overline{\psi(g)}$

$x \in G = HT$
 $x = ht, \quad h \in H, t \in T$
 $|G| = |H||T|$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{x \in G} \frac{1}{|H|} \hat{\chi}(xgx^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \overline{\psi(g)} \quad \leftarrow htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(htgt^{-1}h^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$|H|$ identical terms for $h \in H$

$$= \frac{1}{|G|} \sum_{g \in G} \sum_{t \in T} \hat{\chi}(tgt^{-1}) \overline{\psi(g)}$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{g \in G} \hat{\chi}(tgt^{-1}) \overline{\psi(g)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \left(\sum_{u \in G} \hat{\chi}(u) \overline{\psi(t^{-1}ut)} \right)$$

$$= \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(t^{-1}ht)} = \frac{1}{|G|} \sum_{t \in T} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

For each $t \in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$$= \frac{1}{|G|} |T| \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= \frac{1}{|H|} \sum_{h \in H} \chi(h) \overline{\psi(h)}$$

$$= [\chi, \psi_H]_H \quad \square$$

If ψ is a class function on G then so is $\bar{\psi}$ where $\bar{\psi}(g) = \overline{\psi(g)} = \psi(g^{-1})$.
 Indeed if ψ is a character, $\psi(g) = \text{tr } \pi(g)$, $\pi: G \rightarrow GL_n(\mathbb{C})$
 $\pi(g) \sim$ similar to $\begin{bmatrix} \lambda_1 & * \\ 0 & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix}$ in the case of finite groups) homomorphism

$\lambda_1, \dots, \lambda_n$ are m^{th} roots of unity where $m = \text{exponent of } G = \text{lcm}(|g| : g \in G)$
 $\bar{\lambda}_i = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\bar{\lambda}_i \lambda_i = |\lambda_i|^2 = 1$
 $|\lambda_i|^m = 1$
 $|\lambda_i| = 1$

$$\psi(g) = \sum \lambda_i$$

$$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \bar{\lambda}_i = \overline{\psi(g)}$$

For every finite group G , the irreducible representations of G can be chosen to be unitary.

$$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}, \quad A^* = \bar{A}^T = \overline{A^T}$$

eg. for S_3 , we have an irreducible representation of degree 2.

$\pi_3: S_3 \rightarrow GL_2(\mathbb{C})$	$\chi(g)$
$() \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2
$(12) \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	0
$(123) \mapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$	-1
$(132) \mapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$	-1
$(13) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	0
$(23) \mapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}$	0

This representation uses integer matrix entries.

left-to-right composition

$$(13) = (123)(12)(132)$$

Character values:

$$\chi(g) = \text{tr } \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for \mathbb{C}^2 yields an equivalent representation using unitary matrices:

	$\chi(g)$
$() \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	2
$(123) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}$	-1
$(132) \mapsto \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix}$	-1
$(12) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	0
$(13) \mapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix}$	0
$(23) \mapsto \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}$	0

$\omega = e^{2\pi i/3}$
 $\bar{\omega} = \omega^2 = \omega^{-1}$
 root of $x^2 + x + 1$
 $\omega^2 + \omega + 1 = 0$
 $\bar{\omega} + \omega = \omega^2 + \omega = -1$

To prove the claim (that every finite group has its irred. reps. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on \mathbb{C}^n is equivalent (under change of basis) to standard inner product $B(x, y) = \sum x_i \bar{y}_i$.

Any inner product has the form $(x, y) \mapsto x M y^*$ where $x = (x_1, \dots, x_n)$
 $M^* = M$ $n \times n$ matrix Hermitian, $\det M \neq 0$.
 $y^* = \begin{pmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{pmatrix}$

If we perform an invertible change of variables $x \mapsto xA$ (A invertible $n \times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)
 then $B(xA, yA) = (xA)M(yA)^* = x \underbrace{(AMA^*)}_{\text{congruent to } M} y^*$

For all nonsingular Hermitian M , we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on \mathbb{C}^n are equivalent by change of basis.

Given a representation $\pi: G \rightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x, y)$ on \mathbb{C}^n such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

- ✓ $B(x, x) \geq 0$, equality iff $x = 0$;
- ✓ $B(x, y)$ linear in x , conjugate linear in y ; $B(y, x) = \overline{B(x, y)}$;
- ✓ $B(x\pi(g), y\pi(g)) = B(x, y)$.

To obtain the inner product $B(x, y)$:

$$B(x, y) = \sum_{g \in G} (x \pi(g)) (y \pi(g))^* = \sum_{g \in G} x \pi(g) \pi(g)^* y^* = x M y^*$$

$$M = \sum_{g \in G} \pi(g) \pi(g)^* \text{ is Hermitian.}$$

$$B(x, x) = \sum_{g \in G} (x \pi(g)) (x \pi(g))^* = \sum_{g \in G} \|x \pi(g)\|^2 \geq 0$$

$B(x, x) > 0$ unless $x = 0$.

$$B(x \pi(g), y \pi(g)) = \sum_{u \in G} \underbrace{(x \pi(g) \pi(u))}_{\pi(w)} \underbrace{(y \pi(g) \pi(u))^*}_{\pi(w)} = \sum_{w \in G} (x \pi(w)) (y \pi(w))^* = B(x, y)$$

$w = gu$

Suppose $H \leq G$; let T be a right transversal for H in G i.e. a set of right coset representatives. Every $g \in G$ can be uniquely factored as $g = ht$, $h \in H$, $t \in T$. (Lagrange's theorem) $|G| = |H||T|$

G permutes the right cosets of H by right-multiplication:

$$Ht \mapsto Htg = Ht' \text{ for some } t' \in T.$$

This gives a permutation representation of G acting on the right cosets of H . If $d = |T| = [G : H]$ then we have a homomorphism $\pi: G \rightarrow S_d \subset GL_d(\mathbb{C})$ with perm. character $\psi(g) = \text{tr } \pi(g) = \text{no. of fixed points of } \pi(g)$

Theorem $\psi = (\mathbb{1}_H)^G$ i.e. the perm. character is the induced character obtained from $\mathbb{1}_H =$ principal character, $\mathbb{1}_H(h) = 1$ for all $h \in H$, induced up to G .

Proof $\hat{\mathbb{1}}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$

$$(\mathbb{1}_H)^G(g) = \sum_{t \in T} \hat{\mathbb{1}}_H(tgt^{-1}) = \text{number of } tgt^{-1} \quad (t \in T) \text{ which are in } H$$

$$= |\{t \in T : \underbrace{tgt^{-1} \in H}\}|$$

$$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Htg = Ht$$

\iff the coset Ht is fixed under right-multiplication by $g \in G$.

$$= \psi(g).$$

□

From the character table of G , we can see what all normal subgroups of G are.

$G = S_4$

$(C_G(g))$	24	4	8	3	4
g	(1)	(12)	(12)(34)	(123)	(1234)
χ_1	1	1	1	1	1
χ_2	1	-1	1	1	-1
χ_3	2	0	2	-1	0
χ_4	3	-1	-1	0	1
χ_5	3	1	-1	0	-1

$\chi_2 : G \rightarrow \mathbb{C}^\times$ is a homomorphism
 Since $\chi_2(g) = \text{tr } \pi_2(g)$ of degree $\chi_2(1) = 1$.

Its kernel is $\{g \in G : \chi_2(g) = 1\}$
 $= \{ \text{elements conjugate to } (1), (12)(34) \text{ or } (123) \}$
 $= A_4$.

Another normal subgroup: all elements $g \in G$
 s.t. $\chi_3(g) = \chi_3(1) = 2$

Theorem Let $\pi : G \rightarrow GL_n(\mathbb{C})$ be a representation with character $\chi(g) = \text{tr } \pi(g)$.
 (Its degree is $n = \chi(1)$.) The kernel of χ defined by
 $\ker \chi = \ker \pi = \{g \in G : \pi(g) = I\} = \{g \in G : \chi(g) = \chi(1) = n\}$
 is a normal subgroup of G .

Note: π is a homomorphism; but χ is not a homomorphism unless $n=1$.

Proof If $g \in G$ then $\pi(g)$ is similar to $\begin{bmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \ddots \\ & & & \varepsilon_n \end{bmatrix}$, where $\varepsilon_i^m = 1$ for all $i=1, 2, \dots, n$.
 $\begin{bmatrix} \lambda & & \\ 0 & \lambda & \\ & & \ddots \\ & & & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & & \\ 0 & \lambda^k & \\ & & \ddots \\ & & & \lambda^k \end{bmatrix}$
 $\begin{bmatrix} 1 & & \\ 0 & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ & & \ddots \\ & & & 1 \end{bmatrix}$
 $\pi(g)$ is a root of $\chi - 1$.
 m is the exponent of G i.e. $\pi(g)^m = I$ for every $g \in G$.

$$\Rightarrow \chi(g) = \text{tr } \pi(g) = \varepsilon_1 + \dots + \varepsilon_n = n \iff \varepsilon_1 = \dots = \varepsilon_n = 1 \iff \pi(g) = I$$

$$|\varepsilon_i| = 1 \Rightarrow \text{Re } \varepsilon_i \leq 1; \text{ equality iff } \varepsilon_i = 1.$$


For $G = S_4$, there are four normal subgroups and they all arise as $\ker \chi$ for some χ .

$$\ker \chi_5 = \{ () \}$$

$$\ker \chi_3 = \{ (), (12)(34), (13)(24), (14)(23) \}$$

$$\ker \chi_2 = A_4$$

$$\ker \chi_1 = S_4.$$

Recall: Let $H \leq G$. Then H is normal in G ($H \trianglelefteq G$) iff H is a union of conjugacy classes, iff H is an intersection of kernels of irreducible characters.

In this way we "read off" all the normal subgroups of G from the char. table.

eg. $G = \{\pm 1\} \times \{\pm 1\} = \{ (1,1), (1,-1), (-1,1), (-1,-1) \}$ Klein

Char table	$ C_G(g) $			
	\uparrow	\uparrow	\uparrow	\uparrow
g	$(1,1)$	$(1,-1)$	$(-1,1)$	$(-1,-1)$
χ_1	1	1	1	1
χ_2	1	1	-1	-1
χ_3	1	-1	1	-1
χ_4	1	-1	-1	1

Application to Frobenius groups.

Suppose $G \leq S_n$ (G permutes $\{1, 2, \dots, n\}$)

G permutes any set X , $G \leq \text{Sym}(X) = \{\text{permutations of } X\}$

More generally,
 X : any set.
eg. $X = \{1, 2, \dots, n\}$

G is transitive if for all $x, y \in X$, there exists $g \in G$ mapping $x \mapsto y$.

The stabilizer of a point $x \in X$ is $G_x = \{g \in G : g(x) = x\}$.

Of course $G_x \leq G$.

The orbit of $x \in X$ is $x^G = \{g(x) : g \in G\} \subseteq X$.

$$|x^G| = [G : G_x] = \frac{|G|}{|G_x|} \quad \text{or } G(x) \quad \text{(orbit-stabilizer formula)}$$

$x^G = X$ iff G is transitive.

G is a Frobenius group if

(i) G is transitive

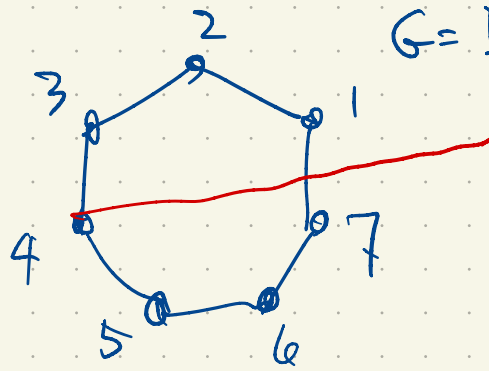
(ii) Point stabilizer is nontrivial

$$\left(|G_x| = \frac{|G|}{|X|} > 1 \right)$$

(iii) the stabilizer of any two points is trivial i.e. $G_x \cap G_y = 1$ for all $x \neq y$ in X .

If m is an odd positive integer then the symmetry group of a regular m -gon (i.e. the dihedral group of order $2m$) is a subgroup of S_m having m rotations, m reflections.

eg. $m=7$



$$G = D_7 = \langle (1234567), (17)(26)(35) \rangle$$

$$G_A = \langle (17)(26)(35) \rangle$$

$()$ = identity in G fixes 7 points;
 nontrivial rotations fix 0 "
 other elements fix 1 point.

This is a Frobenius group.

There are ^{exactly} two groups of order 21: the cyclic group, and a Frobenius group $\langle (1234567), (124)(365) \rangle < S_7$ transitive.

$$\downarrow \text{conjugate by } (124)(365)$$

$$(2461357) = (1357246) = (1234567)^2$$

Eg. $G =$ direct isometries of $\mathbb{R}^2 = \{ \text{translations} \} \vee \{ \text{rotations} \}$
 ↑ (orientation-preserving)

is a Frobenius group. there are lots of finite analogues of this example.

eg. $F = \mathbb{F}_{11} = \{\text{integers mod } 11\}$.

The affine general linear group on F^2 is a subgroup of S_{121} consisting of transformations $v \mapsto Av + b$, $A \in GL_2(F)$, $b \in F^2$.

$GL_2(F) = \{\text{invertible linear transformations on } F^2\}$

$$|GL_2(F)| = (11^2 - 1)(11^2 - 11) = 120 \cdot 110 = 13200$$

$$|AGL_2(F)| = 11^2 \cdot 13200 \quad \text{transitive on } F^2.$$

Stabilizer of $D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in F^2$ is $GL_2(F)$ of order 13200.

This group $AGL_2(F)$ is not a Frobenius group e.g.

$$v \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{fixes all eleven vectors } \begin{bmatrix} a \\ 0 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

The two distinct points $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ are fixed by more than just identity

in fact the subgroup $\begin{bmatrix} 1 & b \\ 0 & c \end{bmatrix}$ $b, c \in F, c \neq 0$
of order 110

Modify the example: $GL_2(F)$ has a subgroup isomorphic to $SL_2(\mathbb{F}_5)$ (order 120), ^{sharply} transitive on the 120 nonzero vectors

The ^{affine linear} transformations on F^2 of the form $v \mapsto Av + b$, $A \in$ subgp of $GL_2(\mathbb{F}_{11})$ isomorphic to $SL_2(\mathbb{F}_5)$, $b \in F^2$,
forms a Frobenius group of order $121 \cdot 120$. Actually this example is sharply 2-transitive

On \mathbb{R}^2 , the transformations $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ which are direct similarities is a Frob. gp., sharply 2-trans (invertible)

Let F be any field. An affine linear map $F \rightarrow F$ is a map $x \mapsto mx+b$, $m \neq 0$.

This group is $AGL_1(F)$.

Now if H is a subgroup of $F^\times = \{\text{nonzero elements of } F\}$ then the affine linear transformations $F \rightarrow F$, $x \mapsto mx+b$, $x \in H, b \in F$ is a Frobenius group. (sharply 2-trans. iff $H = F^\times$).

$$AGL_1(F) \cong \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} : m, b \in F, m \neq 0 \right\} < GL_2(F)$$

$$G = AGL_n(F) \cong \left\{ \left[\begin{array}{c|c} A & * \\ \hline 0 & 1 \end{array} \right] : A \in GL_n(F) \right\} < GL_{n+1}(F).$$

$$\text{has subgroups } K = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & 1 \end{array} \right] : A \in GL_n(F) \right\} \cong GL_n(F)$$

$$H = \left\{ \left[\begin{array}{c|c} I & * \\ \hline 0 & 1 \end{array} \right] \right\} \cong \text{translations of } F^n, \quad x \mapsto x+b$$

$$H \trianglelefteq AGL_n(F)$$

$$K = G_0 = \text{stabilizer of } 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

This is 2-transitive on F^n , not sharply in general.

If we replace $GL_n(F)$ by a subgroup having no nontrivial elements with eigenvalue 1 we get a Frobenius group.

Theorem (Frobenius) Every Frobenius group G has the form

$G = K \rtimes H$ i.e. subgroups H, K satisfy $G = KH$, $K \triangleleft G$, $K \cap H = 1$.

(Note: $KH = \{kh : h \in H, k \in K\}$ is a subgroup assuming one of them is normal.)

$$G/K = KH/K \cong H/(K \cap H) \cong H$$

$$H = G_o \text{ where } o \in X \text{ any point in } X \\ = \{g \in G : g(o) = o\}$$

What is K ? $K = \{1\} \cup \{\text{elements of } G \text{ which don't fix any point}\}$
 $= \{1\} \cup (G - \bigcup_{g \in G} (gHg^{-1})) \subseteq G$.

Since G is transitive, the stabilizer of any point $x \in X$ is conjugate to $H = G_o$. Why? Every point $x \in X$ has the form $x = g(o)$ for some $g \in G$.

If $h \in G_o = H$ then $(ghg^{-1})(x) = (ghg^{-1})(g(o)) = gh(o) = \underbrace{g(o)}_{o = g^{-1}(x)} = x$
 $h(o) = o$

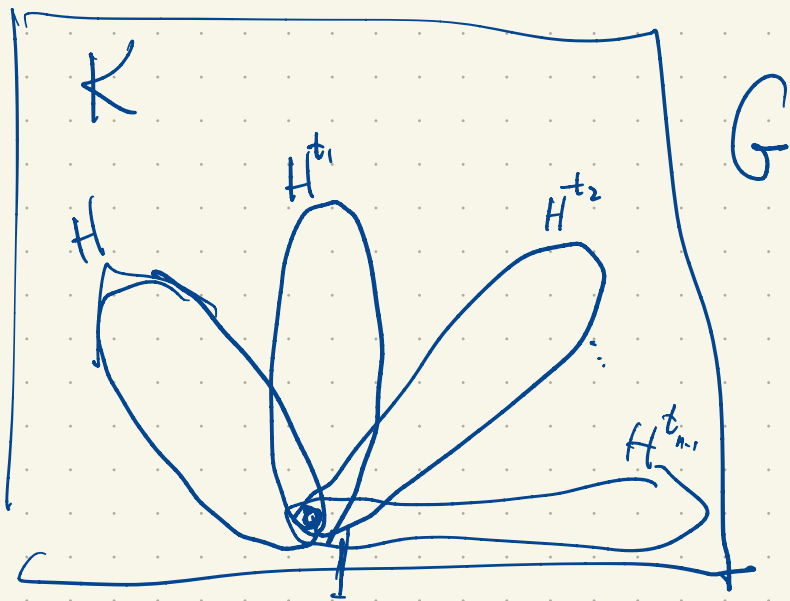
i.e. $ghg^{-1} \in G_x$. So $gHg^{-1} \subseteq G_x$ i.e. $gG_o g^{-1} \subseteq G_x$ $G_o \subseteq g^{-1}G_x g$
 $G_o \supseteq g^{-1}G_x g$

If $f \in G_x$ i.e. $f(x) = x$ then $(g^{-1}fg)(o) = g^{-1}f(x) = g^{-1}x = o$

The problem is: the subset K defined above needs to be a subgroup.
If I can show this, normality follows immediately.

In S_5 (left-to-right),

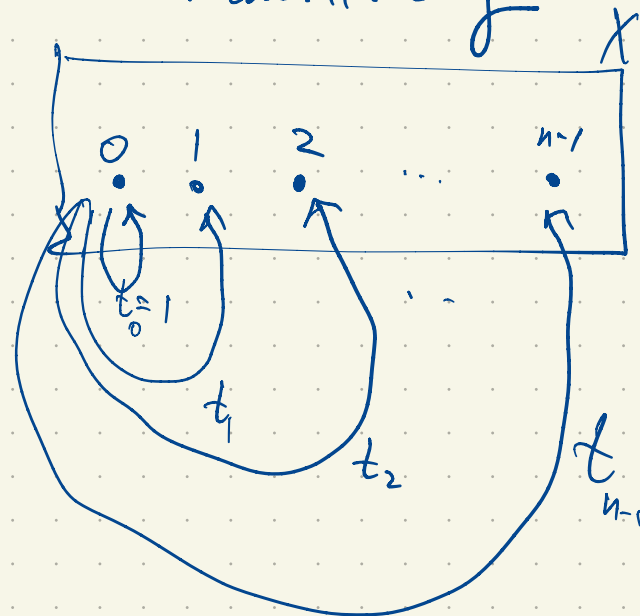
$$(12345)(15342) = (243)$$



$$H = G_0 = \{g \in G : g(0) = 0\}$$

$$0 \in X$$

G permutes X transitively



Any two stabilizers $t_i^{-1} H t_i$, $t_j^{-1} H t_j$ are disjoint for $i \neq j$:

$$H^{t_i} \cap H^{t_j} = 1 \text{ for } i \neq j$$

$$G_i = \{g \in G : g(i) = i\} = t_i^{-1} H t_i$$

$$T = \{t_0, t_1, \dots, t_{n-1}\}$$

right transversal for H in G

Every conjugate of H has $|H| - 1 = \frac{|G|}{n} - 1$ non-identity elements.

$$K = \{1\} \cup \left(G - \bigcup_{i=0}^{n-1} H^{t_i} \right)$$

$$G = \bigsqcup_{i=0}^{n-1} H t_i$$

$$n|H| = |G|$$

$$|H| = \frac{|G|}{n}$$

$$|K| = 1 + \left(|G| - n \left(\frac{|G|}{n} - 1 \right) \right)$$

$$= 1 + |G| - 1 - |G| + n = n$$

Why is K a subgroup of G ? Normality is easy.
 We show that K is an intersection of kernels of characters of G .

Lemma Let θ be a class function on H . Then $\theta^G|_H = \theta$ i.e. $\theta^G(h) = \theta(h)$ for all $h \in H$.
 Such that $\theta(1) = 0$.

Proof $\theta^G(g) = \sum_{i=0}^{n-1} \hat{\theta}(t_i^{-1}gt_i)$ where $\hat{\theta}(g) = \begin{cases} \theta(g), & \text{if } g \in H; \\ 0, & \text{if } g \notin H. \end{cases}$

For $g=1$, $\theta^G(1) = \sum_i \hat{\theta}(1) = \sum_i \theta(1) = 0 = \theta(1)$
 \uparrow
 $t_i^{-1}1t_i = 1$

For $h \in H$, $1 \neq h$, $\theta^G(h) = \sum_i \hat{\theta}(t_i^{-1}ht_i)$
 $= \hat{\theta}(h) = \theta(h)$
 as required. □

When is $t_i^{-1}ht_i \in H$?

i.e. $t_i^{-1}ht_i \in H \cap t_i^{-1}Ht_i$

Only if $i=0$, $t_0=1$, $t_0^{-1}ht_0 = h$

Now let ψ be any irreducible character of H , $\psi \neq \frac{1}{|H|} \mathbf{1}_H$ principal character $\frac{1}{|H|} \mathbf{1}_H(h) = 1$.

Define $\theta(h) = \psi(h) - \psi(1)$ i.e. $\theta = \psi - \psi(1)\mathbf{1}_H$.

So θ is a class function on H with $\theta(1) = \psi(1) - \psi(1) = 0$.

This yields an induced class function θ^G on G satisfying $\theta^G(h) = \theta(h) = \psi(h) - \psi(1)$ for all $h \in H$.

Define $\psi^* = \theta^G + \psi(1)\mathbf{1}_G$. This is a class function on G .

$\psi^*(g) = \theta^G(g) + \psi(1)$

We will prove that this is an irred. char. of G , and furthermore $\bigcap_{\psi \in \text{Irr}_H} \ker \psi^* = K$.

$$\psi \in \text{Irr}_H, \quad \psi \neq 1_H.$$

$$\theta = \psi - \psi(1)1_H$$

$$\begin{aligned} [\theta, \theta]_H &= [\psi - \psi(1)1_H, \psi - \psi(1)1_H]_H \\ &= [\psi, \psi]_H - [\psi, \psi(1)1_H]_H - [\psi(1)1_H, \psi]_H + [\psi(1)1_H, \psi(1)1_H]_H \\ &= 1 - \underbrace{\psi(1)}_0 [\psi, 1_H]_H - \psi(1) \underbrace{[1_H, \psi]_H}_0 + \psi(1)^2 \underbrace{[1_H, 1_H]_H}_1 \\ &= 1 + \psi(1)^2 \end{aligned}$$

Frobenius reciprocity: $[\theta^G, \rho]_G = [\theta, \rho|_H]_H$
 θ class function on H
 ρ " " " G

$$\begin{aligned} [\psi^*, \psi^*]_G &= [\theta^G + \psi(1)1_G, \theta^G + \psi(1)1_G]_G = [\theta^G, \theta^G]_G + \psi(1)[\theta^G, 1_G]_G + \psi(1)[1_G, \theta^G]_G + \psi(1)^2[1_G, 1_G]_G \\ &= \underbrace{[\theta, \theta^G]_H}_{[\theta, \theta]_H} + \psi(1)[\theta, 1_H]_H + \psi(1)[1_H, \theta]_H + \psi(1)^2 \\ &= 1 + \psi(1)^2 + \underbrace{[\psi - \psi(1)1_H, 1_H]_H}_{0} + \underbrace{[\psi(1)1_H, \psi - \psi(1)1_H]_H}_{-\psi(1)^2} \\ &= 1. \end{aligned}$$

$$\psi^* = \sum a_i \chi_i, \quad a_i \in \mathbb{Z}, \quad \chi_i \in \text{Irr } G. \quad - \psi(1)^2$$

$$= \theta^G + \psi(1)1_G$$

$$\psi^*(1) = \theta^G(1) + \psi(1) = \cancel{\theta(1)} + \psi(1) = \psi(1) \in \{1, 2, \dots\}$$

(a pos. integer)
 $\Rightarrow \psi^* \in \text{Irr } G.$

$$\theta^G = \psi^G - \psi(1)1_G$$

$$[\psi^*, \psi^*]_G = \sum a_i^2 = 1 \Rightarrow \text{one of } a_i = \pm 1 \text{ and all other terms are zero.}$$

$$\Rightarrow \psi^* = \pm (\text{irred. char. of } G)$$

Given a finite group G , where do irreducible representations of G come from?

It turns out that they can all be found "inside" the group algebra $\mathbb{C}G$.

An algebra is (usually) is a ring which is also a vector space.

eg. $M(n, \mathbb{C}) = \{n \times n \text{ complex matrices}\}$ is an algebra of dimension n^2 over \mathbb{C} .

An algebra A over a field F has three basic binary operations:

Elements of A can be thought of as "vectors"; elements of F are "scalars".

in F :
scalar + scalar = scalar
scalar \times scalar = scalar

in A :
vector + vector = vector
vector \times vector = vector
scalar \times vector = vector

These give the ring structure for A .

Axioms:

$$(ab)v = a(bv) \quad a, b \in F; \quad v \in A$$

$$a(vw) = (av)w \quad a \in F, \quad v, w \in A$$

$$(uv)w = u(vw) \quad u, v, w \in A$$

These make A into a vector space.

In general, our algebras need not be commutative i.e. as a ring i.e. $vw \neq wv$ in general

Our algebras will have identity 1_A , $1_A v = v$ $1_F = 1 \in F$ scalar identity ($v, w \in A$)

Consider $M(n, \mathbb{C})$: $n \times n$ complex matrices. Noncommutative algebra with identity $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

If A_1, A_2, \dots, A_k are algebras over \mathbb{C} then so is $\bigoplus_{i=1}^k A_i = A_1 \oplus \dots \oplus A_k$ with componentwise operations

$M(n_1, \mathbb{C}) \oplus M(n_2, \mathbb{C}) \oplus \dots \oplus M(n_k, \mathbb{C})$ is an algebra of dimension $\sum_{i=1}^k n_i^2 = n_1^2 + \dots + n_k^2$

$\cong \left\{ \begin{bmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \\ & & & A_k \end{bmatrix} : A_i \in M(n_i, \mathbb{C}) \right\} \subseteq M(n, \mathbb{C}), \quad n = \sum n_i$
subalgebra

Group algebra: Let G be a finite group.
 The group algebra $\mathbb{C}G$ is the set of (formal) linear combinations of group elements (symbolic)

$$\mathbb{C}G = \left\{ \sum_{g \in G} a_g g : a_g \in \mathbb{C} \right\}$$

$$\mathbb{C}S_3 = \left\{ a_1(1) + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) : a_1, \dots, a_6 \in \mathbb{C} \right\}$$

$$\begin{aligned} (5(1) + 2(12) - 7(13)) (4(12) + 5(13)) &= 20(12) + 25(13) + 8(1) + 10(123) - 28(132) \\ &= -27(1) + 20(12) + 25(13) + 10(123) - 28(132) \end{aligned}$$

$\mathbb{C}G$ is an algebra of dimension $|G|$.

$$\mathbb{C}S_3 \cong M(1, \mathbb{C}) \oplus M(1, \mathbb{C}) \oplus M(2, \mathbb{C}) \cong \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{bmatrix} : a, b, c, d, e, f \in \mathbb{C} \right\}$$

These are really the three irreducible reps of S_3 .

$$\pi: S_3 \rightarrow M(4, \mathbb{C})$$

$$\pi(g) = \begin{bmatrix} \pi_1(g) & 0 & 0 & 0 \\ 0 & \pi_2(g) & 0 & 0 \\ 0 & 0 & \pi_3(g) & 0 \\ 0 & 0 & 0 & \pi_3(g) \end{bmatrix}$$

Representation-theoretic terminology	Module terminology
vector space over F with representation of G	FG -module
invariant subspace	submodule
irreducible representation	simple module
completely reducible	semisimple
equivalent representations	isomorphic modules
G -equivariant linear transformation	module homomorphism
trivial representation $g \mapsto (1)$	trivial FG -module F

TABLE 2.2: Glossary

Let A be an algebra over a field F . (ring + vector space over F)
 (think of examples: $M(n, \mathbb{C})$, $\mathbb{C}G$)

Let M be an additive abelian group (usually a vector space over \mathbb{C}).
 We say A acts on M (M is a module for A) if we have binary operation

$$A \times M \rightarrow M \quad \text{Such that} \quad \begin{aligned} (a+a')m &= am + a'm \\ a(m+m') &= am + am' \\ (aa')m &= a(a'm) \\ 1_A m &= m \end{aligned} \quad \begin{aligned} a, a' \in A; \quad m, m' \in M \\ 1_A \in A \text{ identity} \end{aligned}$$

Examples: $A=F$ a field.
 M is a module over F \Leftrightarrow M is a vector space over F .

Eg. A any algebra.
 Then A can act on itself ($M=A$). This is the regular A -module.

A module over A is an A -module.

Eg. $A = M(n, \mathbb{C})$ algebra acting on $M = \mathbb{C}^n$. So avoid thinking of elements of M as being bigger than elements of A .

Eg. $\pi: G \rightarrow GL(V) = \{ \text{invertible linear transformations } V \rightarrow V \}$ homomorphism
 $GL(\mathbb{C}^n) = GL_n(\mathbb{C})$ i.e. representation of G .
 gives rise to a $\mathbb{C}G$ -module V . We have an action of the entire group algebra $\mathbb{C}G$ on V .
 For us, G is a finite group.

If $\alpha = \sum_{g \in G} a_g g \in \mathbb{C}G$ then we extend π to a representation of $A = \mathbb{C}G$

i.e. a homomorphism of algebras $\pi: \mathbb{C}G \rightarrow M(n, \mathbb{C})$
 $\pi(\alpha) = \pi\left(\sum_g a_g g\right) = \sum_g a_g \pi(g) \in M(n, \mathbb{C}) \quad (a_g \in \mathbb{C})$

If A and B are algebras over \mathbb{C} then a homomorphism $A \rightarrow B$ is

a map $f: A \rightarrow B$ such that

$$\begin{cases} f(ca + c'a') = cf(a) + c'f(a') \in B \\ f(aa') = f(a)f(a') \end{cases} \quad \text{for all } c, c' \in \mathbb{C} \\ a, a' \in A$$

$\text{Hom}_{\mathbb{C}}(A, B) = \{ \text{algebra homomorphisms } A \rightarrow B \}$ is a vector space over \mathbb{C}

$\text{End}_{\mathbb{C}}(A) = \text{Hom}_{\mathbb{C}}(A, A) = \{ \text{endomorphisms } A \rightarrow A \}$ is an algebra over \mathbb{C}
into

$$\text{End}_{\mathbb{C}}(\mathbb{C}^n) \cong M(n, \mathbb{C})$$

Suppose M, M' are modules over A .

A map $f: M \rightarrow M'$ is \mathbb{C} -linear ($f \in \text{Hom}_{\mathbb{C}}(M, M')$) if $f(cm + c'm') = cf(m) + c'f(m')$
for all $c, c' \in \mathbb{C}$,
 $m, m' \in M$.

We say $f: M \rightarrow M'$ is A -linear ($f \in \text{Hom}_A(M, M')$) if $f(am + a'm') = af(m) + a'f(m')$.
 A -homomorphism $M \rightarrow M$

f is A -linear $\Leftrightarrow f$ is \mathbb{C} -linear.

Look at $A = \mathbb{C}G$ group algebra over \mathbb{C} .

Modules over $A = \mathbb{C}G$ are the same thing as vector spaces \mathbb{C}^n over \mathbb{C} having a
group action i.e. $\pi: G \rightarrow \text{GL}_n(\mathbb{C})$.
for $v \in \mathbb{C}^n$, $g \in G$, $gv = \pi(g)v$

Let's consider $A = M(n, \mathbb{C})$ as a prelude to $\mathbb{C}G$.

Think of A acting on itself by left-multiplication, acting on $M = M(n, \mathbb{C})$.

What are the $\left\{ \begin{array}{l} \text{left ideals} \\ \text{right ideals} \\ \text{two-sided ideals} \end{array} \right\}$ of $M(n, \mathbb{C})$?

$\left\{ \begin{bmatrix} * & & \\ \vdots & & \\ * & & \end{bmatrix} \begin{bmatrix} \circ & & \\ & & \\ & & \end{bmatrix} \right\}$ is a left ideal. (a submodule of the regular module)

$$\begin{matrix} k \times l & l \times n \\ \uparrow & \uparrow \\ B & [v_1, \dots, v_n] \\ \uparrow & \uparrow \\ & \text{column vectors} \end{matrix} = [Bv_1, Bv_2, \dots, Bv_n]$$

$$B[v_1, 0, \dots, 0] = [Bv_1, 0, \dots, 0]$$

$V_1 = \begin{bmatrix} * & & \\ \vdots & & \\ * & & \end{bmatrix} \begin{bmatrix} \circ & & \\ & & \\ & & \end{bmatrix} \subseteq M(n, \mathbb{C})$ is a left ideal i.e. submodule of the regular A -module $(A = M(n, \mathbb{C}))$ isomorphic to the A -module \mathbb{C}^n .

$\mathbb{C}^n \cong_A V_1$ i.e. stronger than isomorphism as vector spaces over \mathbb{C} $V_1 \cong_{\mathbb{C}} \mathbb{C}^n$

$$\downarrow \phi \quad \uparrow \cong$$

$$v_i \mapsto [v_i, 0, 0, \dots, 0]$$

$$V_i \cong_A \mathbb{C}^n \text{ (iso. as } A\text{-modules)}$$

$$\phi(cv_i + c'v_i') = c\phi(v_i) + c'\phi(v_i') \quad c, c' \in \mathbb{C}; \quad v_i, v_i' \in \mathbb{C}^n \quad \mathbb{C}\text{-linear}$$

$$\phi(Bv_i) = B\phi(v_i) \text{ for all } B \in A = M(n, \mathbb{C}) \quad \text{i.e. } \phi \text{ is } A\text{-equivariant}$$

ϕ preserves the action of A .

The regular A -module is a direct sum of simple A -modules

$$A = V_1 \oplus V_2 \oplus \dots \oplus V_n \quad \text{where } V_1 = \begin{bmatrix} * & & \\ \vdots & & \\ * & & \end{bmatrix} \begin{bmatrix} \circ & & \\ & & \\ & & \end{bmatrix}, \quad V_2 = \begin{bmatrix} \circ & * & \\ \circ & \circ & \\ \vdots & \vdots & \\ \circ & \circ & * \end{bmatrix} \begin{bmatrix} \circ & & \\ & & \\ & & \end{bmatrix}, \quad \dots, \quad V_n = \begin{bmatrix} \circ & & \\ & & \\ \circ & & * \\ & & \vdots \\ & & \circ \end{bmatrix} \begin{bmatrix} \circ & & \\ & & \\ & & \end{bmatrix}$$

$\dim A = n^2, \quad \dim V_i = n. \quad V_i \subseteq A$ left ideal.
Each V_i is a simple A -module.

$A = M(n, \mathbb{C})$ is transitive on the nonzero column vectors in \mathbb{C}^n .

If $v_0 \in \mathbb{C}^n$ is nonzero then $Av_0 = \mathbb{C}^n$

The left ideal $V_i \subseteq A$ is minimal ideal i.e. simple module i.e. irreducible representation of the algebra A .

A has infinitely many minimal left ideals for $n > 1$

eg. $A = M(3, \mathbb{C})$ has minimal left ideals $V_1 = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$,

$$\left\{ \begin{bmatrix} a & a & 0 \\ b & b & 0 \\ c & c & 0 \end{bmatrix} : a, b, c \in \mathbb{C} \right\} \quad \mathcal{B} [v_1, v_1, 0] = [Bv_1, Bv_1, 0]$$

For every row vector $w = [w_1, w_2, \dots, w_n]$, $w_i \in \mathbb{C}$

$$V_w = \left\{ vw : v \in \mathbb{C}^n \right\}$$

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [w_1, w_2, \dots, w_n]$$

$$\mathcal{B}(vw) = (Bv)w \in V_w$$

There are many ways to decompose $A = \bigoplus_{i=1}^n V_i$ as a direct sum of n minimal simple left ideals.

But all the minimal left ideals of the regular module are isomorphic to \mathbb{C}^n as A -modules.

The only 2-sided ideals of A are 0 and A .

A is a simple algebra.

Let M be an A -module. M is simple if its only submodules are 0 and M .

eg. $A = M(n, \mathbb{C})$, \mathbb{C}^n is a simple module.

M is semisimple if M is a direct sum of simple modules.

\iff every submodule of M has a complementary submodule.

(PT)

Stick to finite dimensional

For G any finite group, $A = \mathbb{C}G$ group algebra, every (finite dimensional) module for A is semisimple.

(module/representation for G) This is Maschke's theorem, which we have proved. (p. 56 of handout)

Every irreducible representation of G is isomorphic to a minimal left ideal of $A = \mathbb{C}G$.

Theorem (p. 6) Let R be a semisimple algebra. Then M is isomorphic to a minimal left ideal $I \subseteq R$.
 Let M be a simple R -module. Then M is isomorphic to a minimal left ideal $I \subseteq R$.
 (minimal: (left module) no nonzero submodules)

Proof Let $v \in M$ be nonzero. Then $Rv \subseteq M$ is a nonzero submodule. ($R \cdot Rv \subseteq Rv$)

Since M is simple, $Rv = M$. Let $\phi: R \rightarrow M$, $r \mapsto rv$. This is a homomorphism of R -modules. (If $s \in R$ then $\phi(sr) = srv = s(rv) = s\phi(r)$.)

The annihilator of v in R is $\ker \phi = \{r \in R : rv = 0\} \subseteq R$ is an ideal since it's the kernel of a homomorphism. (so $\ker \phi$ is a submodule of the regular module).

So $R = \ker \phi \oplus J$ for some ideal $J \subseteq R$. The First Isomorphism theorem yields $M = \phi(R) = Rv = R/\ker \phi \cong J$. \square

In the case $R = \mathbb{C}G$, $|G| < \infty$, $\pi: G \rightarrow \text{GL}_n(\mathbb{C})$ irreducible representation, then $M = \mathbb{C}^n$ is a module for R . $\left(\sum_{g \in G} a_g g \right) v = \sum_{g \in G} a_g \pi(g)v$.

We now know that π comes from some minimal left ideal of $R = \mathbb{C}G$.

Why are there only finitely many irred. reps. of G up to equivalence?

There are infinitely many left ideals of $\mathbb{C}G$ in general, but only finitely many up to isomorphism.

Theorem (p.7) Let R be semisimple, so $R = I_1 \oplus I_2 \oplus \dots \oplus I_m$ where each I_i is a minimal left ideal of R . Then every simple R -module is isomorphic (as an R -module) to I_i for some $i \in \{1, 2, \dots, m\}$. In particular, R has only finitely many simple modules up to isomorphism.

(Every finite group has only a finite number of irred. reps. up to equivalence)

Proof Let M be a simple R -module. By the previous result, WLOG M is a minimal left ideal of R . So $R = M \oplus J$ for some left ideal J . J is a proper ideal ($J \subset R$, $J \neq R$) so there exists $i \in \{1, 2, \dots, m\}$ such that $I_i \not\subset J$. Claim: $M \cong I_i$ as R -modules.

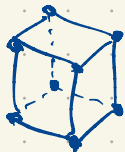
$$I_i \cong I_i / \underbrace{(I_i \cap J)}_{\substack{\text{proper sub-ideal of } I_i \\ \text{hence } I_i \cap J = 0}} \cong (J + I_i) / J \subseteq \underbrace{R/J}_{\substack{\text{Second Isomorphism} \\ \text{Theorem}}} \cong M.$$

must be equality since M is minimal. □

Let's take another detour: application to spectra of Cayley graphs.

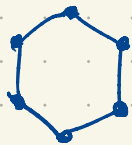
L. Babai

Eg.



$G = \mathbb{F}_2^3$ under addition
 $S = \{(100), (010), (001)\}$

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$



$G = S_3$
 $S = \{(12), (23)\}$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

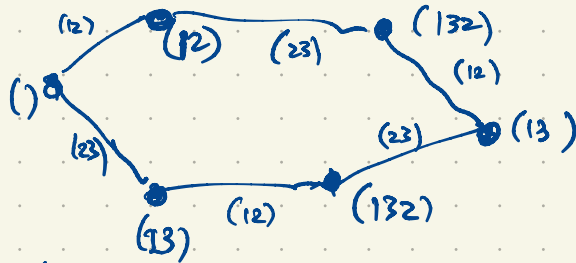
Let G be a group and $S \subseteq G$ a generating set (i.e. $\langle S \rangle = G$) such that $1 \notin S$; $s \in S \iff s^{-1} \in S$.

The Cayley graph $\text{Cay}(G, S)$ has vertex set G .

There is an edge between g and h ($g, h \in G$) iff there exists $s \in S$ such that $g = hs$ i.e. $h^{-1}g \in S$.

$\text{Cay}(G, S)$ is a simple graph (undirected graph with no loops or multiple edges).

$$\text{Cay}(S_3, \{(12), (23)\}) =$$



$\text{Cay}(G, S)$ is a regular graph of degree $|S|$ with $|G|$ vertices.

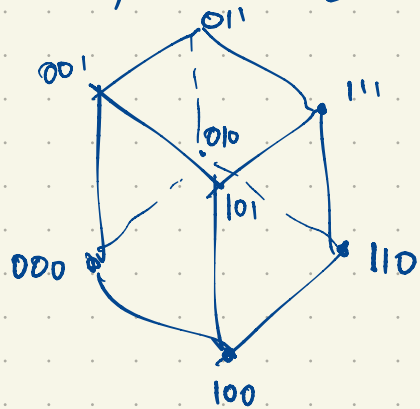
It is vertex-transitive since G acts on $\text{Cay}(G, S)$ by left-multiplication

$$x \sim y \iff y = xs \iff gx = gys \iff gx \sim gy$$

for some $s \in S$

But transitive graphs don't have to be Cayley graphs.

$$G = \mathbb{F}_2^3, \quad F = \mathbb{F}_2 = \{0, 1\}, \quad S = \{(100), (010), (001)\}$$



G has 8 irreducible characters, all linear (i.e. degree 1) indexed by $v \in G$.

$$\chi_v(u) = (-1)^{u \cdot v}$$

Given an abelian group G , $|G| = n$, there are n irred. characters.

$S \subset G$ generators as above

The eigenvalues of the adjacency matrix $\lambda_1, \dots, \lambda_n$ are

$$\lambda_i = \sum_{s \in S} \chi_i(s)$$

$$\text{Irr } G = \{ \chi_1, \chi_2, \dots, \chi_n \}, \quad \chi_i(1) = 1.$$

For the Hamming cube on 8 vertices, as above,

$$\lambda_v = \chi_v((100)) + \chi_v((010)) + \chi_v((001)) = (-1)^{v \cdot (100)} + (-1)^{v \cdot (010)} + (-1)^{v \cdot (001)}$$

$$v \in \mathbb{F}_2^3$$

$$\lambda_{000} = (-1)^0 + (-1)^0 + (-1)^0 = 1 + 1 + 1 = 3$$

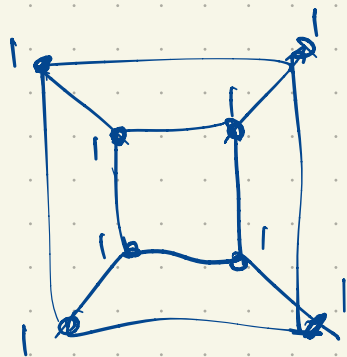
$$\lambda_{100} = (-1)^1 + (-1)^0 + (-1)^0 = -1 + 1 + 1 = 1 = \lambda_{010} = \lambda_{001}$$

$$\lambda_{110} = (-1)^1 + (-1)^1 + (-1)^0 = -1 + (-1) + 1 = -1 = \lambda_{101} = \lambda_{011}$$

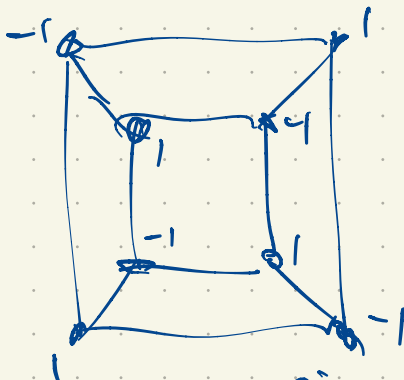
$$\lambda_{111} = -1 - 1 - 1 = -3$$

$$CG \rightarrow CS$$

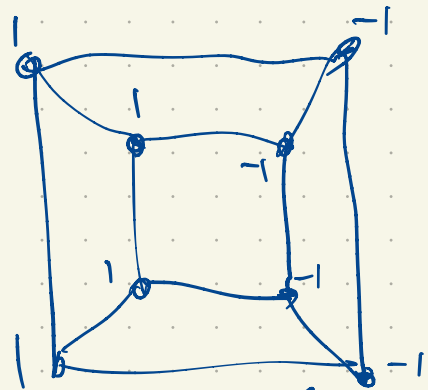
multiply by $\sigma = \sum_{s \in S} s$



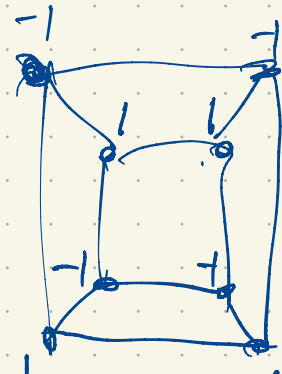
eigenvector with multiplicity 1



eigenvalue 3



eigenvalue 1
(multiplicity 3)



eigenvalue -1
(multiplicity 3)

$$G = S_3$$

$ G $	1	2	3
g	(1)	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Let G be any finite gp with $|G| = n = n_1^2 + n_2^2 + \dots + n_k^2$

$k =$ no. of conjugacy classes in G

$$\text{Irr } G = \{\chi_1, \dots, \chi_k\}, \quad \chi_i(1) = n_i$$

The adjacency matrix of $\text{Cay}(G, S)$ is $n \times n$ diagonalizable with n real eigenvalues

$$\lambda_{i,1}, \dots, \lambda_{i,n_i} \text{ each with multiplicity } n_i; \quad 1 \leq i \leq k$$

$$\lambda_{i,1}^t + \lambda_{i,2}^t + \dots + \lambda_{i,n_i}^t = \sum_{s_1, \dots, s_t \in S} \chi_i(s_1 s_2 \dots s_t), \quad 1 \leq i \leq k$$

$$1 \leq t \leq n_i$$

Eg. $G = S_3, S = \{(12), (23)\}$

$i=1$ gives $\lambda_1 = \lambda_{1,1}$, $\lambda_1 = \sum_{s \in S} \chi_1(s) = \chi_1((12)) + \chi_1((23)) = 1 + 1 = 2$ with multiplicity $n_1 = 1$.

$i=2$ gives $\lambda_2 = \lambda_{2,1}$, $\lambda_2 = \sum_{s \in S} \chi_2(s) = \chi_2((12)) + \chi_2((23)) = -1 - 1 = -2$ with multiplicity $n_2 = 1$.

$i=3$ gives $\lambda_{3,1}, \lambda_{3,2}$

$(t=1) \quad \lambda_{3,1} + \lambda_{3,2} = \sum_{s \in S} \chi_3(s) = \chi_3((12)) + \chi_3((23)) = 0 + 0 = 0$

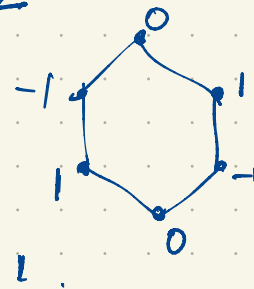
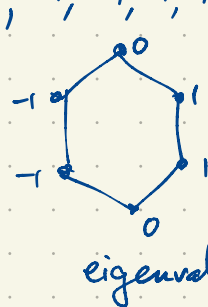
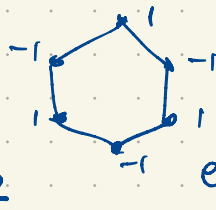
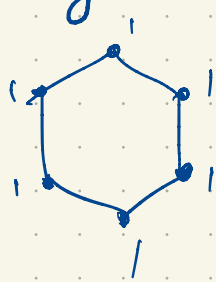
$(t=2) \quad \lambda_{3,1}^2 + \lambda_{3,2}^2 = \sum_{s_1, s_2 \in S} \chi_3(s_1 s_2) = \chi_3(\underbrace{(12)(12)}_1) + \chi_3(\underbrace{(12)(23)}_{(132)}) + \chi_3(\underbrace{(23)(12)}_{(132)}) + \chi_3(\underbrace{(23)(23)}_1)$

$$\lambda_{3,1} = \lambda, \quad \lambda_{3,2} = -\lambda$$

$$\lambda^2 + \lambda^2 = 2\lambda^2 = 2 \Rightarrow \lambda = \pm 1$$

gives $1, -1$ both with multiplicity $n_3 = 2$.

In summary the 6-cycle has spectrum $-2, -1, -1, 1, 1, 2$



For each $i \in \{1, 2, \dots, k\}$, we solve n_i equations (one for each $t \in \{1, \dots, n_i\}$)

in n_i unknowns $\lambda_{i,j}$, $j \in \{1, 2, \dots, n_i\}$.

In general the n_i symmetric polynomials $\lambda_{i,1}^t + \dots + \lambda_{i,n_i}^t$, $1 \leq t \leq n_i$

Use
(Newton's identities)

can be re-expressed in terms of the elementary symmetric polynomials $e_j = e_j(\lambda_{i,1}, \dots, \lambda_{i,n_i})$ which are the coefficients of

$$(x + \lambda_{i,1})(x + \lambda_{i,2}) \dots (x + \lambda_{i,n_i}) = x^{n_i} + e_1 x^{n_i-1} + e_2 x^{n_i-2} + \dots + e_{n_i} x + e_n$$

i.e. $e_n = \lambda_{i,1} \lambda_{i,2} \dots \lambda_{i,n_i}$

$$e_2 = \sum \lambda_{i,r} \lambda_{i,s}$$

$$e_1 = \lambda_{i,1} + \dots + \lambda_{i,n_i}$$