Math 5555

# Abstract Algebra II

Book 2

Induction takes class functions on $H$ to class functions on $G$
characters ⋯ characters on $G$
representations of $H$ ⋯ representations of $G$.

Let $\chi$ be a class function on $H \le G$ i.e. $\chi : H \to \mathbb{C}$, $\chi(xhx^{-1}) = \chi(h)$ for all $x, h \in H$.
To get a class function on $G$, start with the trivial extension

$$\hat{\chi} : G \longrightarrow \mathbb{C}$$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H; \\ 0 & \text{if } g \notin H. \end{cases}$$

To make this into a class function, use an averaging over conjugates as we did before. This leads to $\chi^G : G \longrightarrow \mathbb{C}$ :

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1}) \qquad (\chi^G \text{ is } \underline{\text{induced}} \text{ from } \chi, \quad \chi^G = \mathrm{Ind}_H^G \chi )$$

If $u \in G$ then $\chi^G(ugu^{-1}) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xugu^{-1}x^{-1}) = \frac{1}{|H|} \sum_{w \in G} \hat{\chi}(wgw^{-1})$

$w^{-1} = u^{-1}x^{-1}$
$w = xu$
$wu^{-1} = x$

$$= \chi^G(g).$$

So $\chi^G$ is a class function on $G$.

Note: Let $T$ be a set of right coset representatives for $H$ in $G$.
So every element $g \in G$ is uniquely expressible as $g = ht$, $h \in H$, $t \in T$

$$|G| = |H||T| \qquad (\text{Lagrange's Theorem}), \qquad |T| = \frac{|G|}{|H|} = [G : H].$$

$\chi : H \to \mathbb{C}$ is a class function on $H$; $T$ is a right transversal for $H$ in $G$ (set of right coset representatives)

$T = \{t_1, \dots, t_\ell\}$

$G = Ht_1 \sqcup Ht_2 \sqcup \cdots \sqcup Ht_\ell$

$= \bigsqcup_{t \in T} Ht.$

$$\chi^G(g) = \frac{1}{|H|} \sum_{x \in G} \hat{\chi}(xgx^{-1})$$

$x = ht$, $h \in H$, $t \in T$

$$= \frac{1}{|H|} \sum_{t \in T} \sum_{h \in H} \hat{\chi}(htgt^{-1}h^{-1})$$

$htgt^{-1}h^{-1} \in H \iff tgt^{-1} \in H$

$$= \frac{1}{|H|} \sum_{t \in T} \underbrace{\sum_{h \in H} \hat{\chi}(tgt^{-1})}_{|H| \text{ terms equal to } \hat{\chi}(tgt^{-1})}$$

$$= \frac{1}{|H|} \sum_{t \in T} |H| \hat{\chi}(tgt^{-1}) = \sum_{t \in T} \hat{\chi}(tgt^{-1}) = \sum_{i=1}^{\ell} \hat{\chi}(t_i g t_i^{-1})$$

Special case: $\chi = \chi_1 =$ trivial (principal) character of $H$, $\chi(h) = 1$.

$\chi^G$ isn't the principal character of $G$ unless $G = H$.

$G$ permutes the right cosets of $H$ by right multiplication giving a permutation representation $g \in G$ permutes $Ht_i \longmapsto Ht_i g = Ht_j$

$\chi^G = (1_H)^G$ is the perm. character of $G$ acting on cosets of $H$. $\quad G \longrightarrow S_\ell \quad \ell = [G:H] = |T|$.

Eg. Construct the character table of $G = S_4$ making use of the character table of $S_3 = H \le G$.

$H$:

| $\lvert C_H(g)\rvert$ | 6 | 2 | 3 |
|---|---|---|---|
| $g$ | $()$ | $(12)$ | $(123)$ |
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 |
| $\chi_3$ | 2 | 0 | $-1$ |

Check: $[\psi_3, \psi_3] = 1$

$G = S_4$:

| $\lvert C_G(g)\rvert$ | 24 | 4 | 8 | 3 | 4 |
|---|---|---|---|---|---|
| $g$ | $()$ | $(12)$ | $(12)(34)$ | $(123)$ | $(1234)$ |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\psi_3$ | 2 | 0 | 2 | $-1$ | 0 |
| $\psi_4$ | 3 | | | | |
| $\psi_5$ | 3 | | | | |
| $\psi$ | 3 | 1 | 3 | 0 | 1 |

$[\psi_1, \psi_2] = 0$

$\underbrace{\psi(1)}_{3} = \underbrace{\psi_1(1)}_{1} + \underbrace{\psi_3(1)}_{2}$

$\psi = \sum_{i=1}^{5} a_i \psi_i$

$[\psi, \psi]_G = \frac{9}{24} + \frac{1}{4} + \frac{9}{8} + \frac{0}{3} + \frac{1}{4}$

$= \frac{3 + 2 + 9 + 0 + 2}{8} = \frac{16}{8}$

$= 2$

$= \sum_{i=1}^{5} a_i^2 = 1^2 + 1^2 + 0^2 + 0^2 + 0^2$

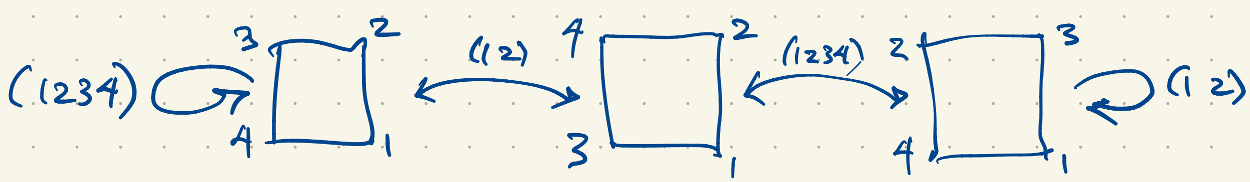
$\psi = \psi_1 + \psi_3$
(degree 1) (degree 2)

$a_1 = [\psi, \psi_1] = \frac{3}{24} + \frac{1}{4} + \frac{3}{8} + \frac{0}{3} + \frac{1}{4} = \frac{1+2+3+0+2}{8} = 1$

$S_4$ has $k = 5$ conjugacy classes
... irreducible representations/characters
of degree $n_1, n_2, \ldots, n_5 \geq 1$, $n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 = |G| = 24$

$S_4$ has normal subgroups
$1, S_4, A_4, K = \langle (12)(34), (13)(24) \rangle = \{ (), (12)(34), (13)(24), (14)(23) \}$.

$S_4$ permutes the conjugacy class of $(12)(34)$ in all $3! = 6$ possible ways
There is a permutation action $S_4 \longrightarrow \mathrm{Sym}\{(12)(34), (13)(24), (14)(23)\} \cong S_3$
with kernel $K$ of order 4.




$C_G((13)(24))$ $C_G((14)(23))$ $C_G((12)(34))$

This gives a permutation representation of $S_4$ of degree 3.
Its permutation character is
$\psi(()) = 3$
$\psi(k) = 3$ for $k \in K$
$\psi((12)) = 1$
$\psi((1234)) = 1$ $\psi((123)) = 0$ $\psi((12)(34)) = 3$

$H = S_3$:

| $\lvert C_H(g)\rvert$ | 6 | 2 | 3 |
|---|---|---|---|
| $g$ | $()$ | $(12)$ | $(123)$ |
| $\chi_1$ | 1 | 1 | 1 |
| $\chi_2$ | 1 | $-1$ | 1 |
| $\chi_3$ | 2 | 0 | $-1$ |

$\chi = \chi_2$

$G = S_4$:

| $\lvert C_G(g)\rvert$ | 24 | 4 | 8 | 3 | 4 |
|---|---|---|---|---|---|
| $g$ | $()$ | $(12)$ | $(12)(34)$ | $(123)$ | $(1234)$ |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\psi_3$ | 2 | 0 | 2 | $-1$ | 0 |
| $\psi_4$ | 3 | $-1$ | $-1$ | 0 | 1 |
| $\psi_5$ | 3 | 1 | $-1$ | 0 | $-1$ |
| $\chi^G$ | 4 | $-2$ | 0 | 1 | 0 |

$$[\chi^G, \chi^G]_G = \frac{16}{24} + \frac{4}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 2$$

$\Rightarrow$ $\chi^G$ has irreducible constituents of multiplicity 1, 1, 0, 0, 0

$$[\chi^G, \psi_1] = \frac{4}{24} - \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 0$$

$$[\chi^G, \psi_2] = \frac{4}{24} + \frac{2}{4} + \frac{0}{8} + \frac{1}{3} + \frac{0}{4} = 1$$

$T = \langle (1234) \rangle = \{(), (1234), (13)(24), (1432)\}$ right transversal for $H$ in $G$

$\chi^G = \chi_2^G$ is a character on $G$

$$\chi^G(g) = \sum_{t \in T} \hat{\chi}(tgt^{-1})$$

where $\hat{\chi} : G \longrightarrow \mathbb{C}$

$$\hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g \in H \\ 0 & \text{if } g \notin H \end{cases}$$

$$\chi^G(1) = 1 + 1 + 1 + 1 = 4$$

$$\chi^G((12)) = \hat{\chi}((12)) + \hat{\chi}((23)) + \cancel{\hat{\chi}((34))} + \cancel{\hat{\chi}((14))}$$
$$= -1 - 1 + 0 + 0 = -2$$

$$\chi^G((12)(34)) = \hat{\chi}((12)(34)) + \hat{\chi}((23)(41)) + \hat{\chi}((34)(12)) + \hat{\chi}((41)(23)) = 0 + 0 + 0 + 0 = 0$$

$$\chi^G((123)) = \hat{\chi}((123)) + \cancel{\hat{\chi}((234))} + \cancel{\hat{\chi}((341))} + \cancel{\hat{\chi}((412))}$$
$$= 1 + 0 + 0 + 0 = 1$$

$$\chi^G = \psi_2 + \psi_4$$
$$\psi_4 = \chi^G - \psi_2$$

$$[\psi_4, \psi_4]_G = \frac{9}{24} + \frac{1}{4} + \frac{1}{8} + \frac{0}{3} + \frac{1}{4} = \frac{3 + 2 + 1 + 0 + 2}{8} = 1$$

**Frobenius Reciprocity** Let $\chi$ be a class function on $H \leq G$ and let $\psi$ be a class function on $G$. Then

$$[\psi_H, \chi]_H = [\psi, \chi^G]_G$$

$\psi_H = \psi|_H$

$$\chi^G(g) = \frac{1}{|H|}\sum_{x\in G} \hat{\chi}(xgx^{-1}) \qquad \hat{\chi}(g) = \begin{cases} \chi(g) & \text{if } g\in H \\ 0 & \text{if } g\notin H \end{cases}$$

**Proof** $[\chi^G, \psi]_G = \frac{1}{|G|}\sum_{g\in G} \chi^G(g)\overline{\psi(g)}$

$= \frac{1}{|G|}\sum_{g\in G}\sum_{x\in G}\frac{1}{|H|}\hat{\chi}(x^{-1}gx)\overline{\psi(g)}$

$x\in G = HT$
$x = ht, \quad h\in H, t\in T$
$|G| = |H||T|$

$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{x\in G}\hat{\chi}(xgx^{-1})\overline{\psi(g)}$

$htgt^{-1}h^{-1}\in H \iff tgt^{-1}\in H$

$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\hat{\chi}(htgt^{-1}h^{-1})\overline{\psi(g)}$

$= \frac{1}{|G||H|}\sum_{g\in G}\sum_{h\in H}\sum_{t\in T}\underbrace{\hat{\chi}(tgt^{-1})\overline{\psi(g)}}$

$|H|$ identical terms ~~for~~ $h\in H$

$= \frac{1}{|G|}\sum_{g\in G}\sum_{t\in T}\hat{\chi}(tgt^{-1})\overline{\psi(g)}$

$= \frac{1}{|G|}\sum_{t\in T}\left(\sum_{g\in G}\hat{\chi}(tgt^{-1})\overline{\psi(g)}\right)$

For each $t\in T$ reparameterize the inner sum, $u = tgt^{-1}$, $g = t^{-1}ut$

$= \frac{1}{|G|}\sum_{t\in T}\left(\sum_{u\in G}\hat{\chi}(u)\overline{\psi(t^{-1}ut)}\right)$

$= \frac{1}{|G|}\sum_{t\in T}\sum_{h\in H}\chi(h)\overline{\psi(t^{-1}ht)} = \frac{1}{|G|}\sum_{t\in T}\sum_{h\in H}\chi(h)\overline{\psi(h)}$

$= \frac{1}{|G|}|T|\sum_{h\in H}\chi(h)\overline{\psi(h)}$

$= \frac{1}{|H|}\sum_{h\in H}\chi(h)\overline{\psi_H(h)}$

$= [\chi, \psi_H]_H$. $\square$

If $\psi$ is a class function on $G$ then so is $\overline{\psi}$ where $\overline{\psi}(g) = \overline{\psi(g)} = \psi(g^{-1})$.

Indeed if $\psi$ is a character, $\psi(g) = \operatorname{tr} \pi(g)$, $\pi : G \to GL_n(\mathbb{C})$ homomorphism

$\pi(g) \underset{\text{similar to}}{\sim} \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ (actually $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ in the case of finite groups)

$\lambda_1, \dots, \lambda_n$ are $m^{th}$ roots of unity where $m =$ exponent of $G = \operatorname{lcm}(|g| : g \in G)$

$\overline{\lambda_i} = \lambda_i^{-1}$ since $\lambda_i^m = 1$, $\overline{\lambda_i}\lambda_i = |\lambda_i|^2 = 1$

$|\lambda_i|^m = 1$

$|\lambda_i| = 1$

$\psi(g) = \sum \lambda_i$

$\psi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda_i} = \overline{\psi(g)}$

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For every finite group $G$, the irreducible representations of $G$ can be chosen to be <u>unitary</u>.

$U_n(\mathbb{C}) = \{A \in \mathbb{C}^{n \times n} : AA^* = A^*A = I\}$, $A^* = \overline{A}^T = \overline{A^T}$.

eg. for $S_3$, we have an irreducible representation of degree 2.

$$\pi_3 : S_3 \to GL_2(\mathbb{C}) \qquad \chi(g)$$

$$() \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad 2$$

$$(12) \longmapsto \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \qquad 0$$

$$(123) \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \qquad -1$$

$$(132) \longmapsto \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \qquad -1$$

$$(13) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 0$$

$$(23) \longmapsto \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \qquad 0$$

This representation uses integer matrix entries.

left-to-right composition

$$(13) = (123)(12)(132)$$

Character values:

$$\chi(g) = \operatorname{tr} \pi(g)$$

A change of basis from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ takes one representation to the other.

A different choice of basis for $\mathbb{C}^2$ yields an equivalent representation using unitary matrices:

$$\chi(g)$$

$$() \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad 2$$

$$(123) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix}, \qquad -1$$

$$(132) \longmapsto \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} \qquad -1$$

$$(12) \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad 0$$

$$(13) \longmapsto \begin{bmatrix} \omega & 0 \\ 0 & \bar{\omega} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{\omega} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{\omega}} & 0 \\ 0 & \omega \end{bmatrix} = \begin{bmatrix} 0 & \bar{\omega} \\ \omega & 0 \end{bmatrix} \qquad 0$$

$$(23) \longmapsto \begin{bmatrix} 0 & \omega \\ \bar{\omega} & 0 \end{bmatrix} \qquad 0$$

$$\omega = e^{2\pi i/3}$$

$$\bar{\omega} = \omega^2 = \omega^{-1}$$

root of $x^2 + x + 1$

$$\omega^2 + \omega + 1 = 0$$

$$\bar{\omega} + \omega = \omega^2 + \omega = -1$$

To prove the claim (that every finite group has its irred. repns. equivalent to equivalent to unitary representations) we use a fact from linear algebra: any inner product on $\mathbb{C}^n$ is equivalent (under change of basis) to standard inner product $B(x,y) = \sum x_i \overline{y_i}$.

Any inner product has the ~~form~~ $(x,y) \longrightarrow xMy^*$ where $x = (x_1, \dots, x_n)$

$M^* = M$ $n \times n$ matrix Hermitian, $\det M \neq 0$. $y^* = \begin{pmatrix} \overline{y_1} \\ \vdots \\ \overline{y_n} \end{pmatrix}$

If we perform an invertible change of variables $x \longmapsto xA$ ($A$ invertible $n \times n$ matrix i.e. $A \in GL_n(\mathbb{C})$)

then $B(xA, yA) = (xA)M(yA)^* = x\underbrace{(AMA^*)}_{\text{congruent to } M}y^*$

For all nonsingular Hermitian $M$, we can find $A \in GL_n(\mathbb{C})$ such that $AMA^* = I$ (using Gram-Schmidt) i.e. any two inner products on $\mathbb{C}^n$ are equivalent by change of basis.

Given a representation $\pi : G \longrightarrow GL_n(\mathbb{C})$, we will find an inner product $B(x,y)$ on $\mathbb{C}^n$ such that all matrices $\pi(g)$, $g \in G$ preserve the inner product:

✓ $B(x,x) \geq 0$, equality iff $x = 0$;

✓ $B(x,y)$ linear in $x$, conjugate linear in $y$; $B(y,x) = \overline{B(x,y)}$;

✓ $B(x\pi(g), y\pi(g)) = B(x,y)$.

To obtain the inner product $B(x,y)$ :

$$B(x,y) = \sum_{g\in G} (x\pi(g))(y\pi(g))^* = \sum_{g\in G} x\pi(g)\pi(g)^* y^* = xMy^*$$

$$M = \sum_{g\in G} \pi(g)\pi(g)^* \quad \text{is Hermitian.}$$

$$B(x,x) = \sum_{g\in G} (x\pi(g))(x\pi(g))^* = \sum_{g\in G} \| x\pi(g)\|^2 \geq 0$$

$B(x,x) > 0$ unless $x = 0$.

$$B(x\pi(g), y\pi(g)) = \sum_{u\in G} (x\underbrace{\pi(g)\pi(u)}_{\pi(w)})(y\underbrace{\pi(g)\pi(u)}_{\pi(w)})^* = \sum_{w\in G} (x\pi(w))(y\pi(w))^*$$

$w = gu$

$$= B(x,y)$$

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Suppose $H \leq G$; let $T$ be a right transversal for $H$ in $G$, i.e. a set of right coset representatives. Every $g\in G$ can be uniquely factored as $g = ht$, $h\in H$, $t\in T$. (Lagrange's theorem) $|G| = |H||T|$

$G$ permutes the right cosets of $H$ by right-multiplication:

$$Ht \longmapsto Htg = Ht' \quad \text{for some } t'\in T.$$

This gives a permutation representation of $G$ acting on the right cosets of $H$. If $d = |T| = [G:H]$ then we have a homomorphism $\pi : G \longrightarrow S_d \subset GL_d(\mathbb{C})$. with perm. character $\psi(g) = \mathrm{tr}\,\pi(g) =$ no. of fixed points of $\pi(g)$

**Theorem** $\psi = (1_H)^G$ i.e. the perm. character is the induced character obtained from $1_H$ = principal character, $1_H(h) = 1$ for all $h \in H$, induced up to $G$.

**Proof** $\hat{1}_H(g) = \begin{cases} 1, & \text{if } g \in H; \\ 0, & \text{if } g \in G, g \notin H. \end{cases}$

$$(1_H)^G(g) = \sum_{t \in T} \hat{1}_H(tgt^{-1}) = \text{number of } tgt^{-1} \ (t \in T) \text{ which are in } H$$

$$= |\{t \in T : tgt^{-1} \in H\}|$$

$tgt^{-1} \in H \iff Htgt^{-1} = H \iff Htg = Ht$
$\iff$ the coset $Ht$ is fixed under right-multiplication by $g \in G$.

$$= \psi(g). \qquad \square$$

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From the character table of $G$, we can see what all normal subgroups of $G$ are.

$G = S_4$:

| $\lvert C_G(g)\rvert$ | 24 | 4 | 8 | 3 | 4 |
|---|---|---|---|---|---|
| $g$ | $()$ | $(12)$ | $(12)(34)$ | $(123)$ | $(1234)$ |
| $\psi_1$ | 1 | 1 | 1 | 1 | 1 |
| $\psi_2$ | 1 | $-1$ | 1 | 1 | $-1$ |
| $\psi_3$ | 2 | 0 | 2 | $-1$ | 0 |
| $\psi_4$ | 3 | $-1$ | $-1$ | 0 | 1 |
| $\psi_5$ | 3 | 1 | $-1$ | 0 | $-1$ |

$\psi_2 : G \to \mathbb{C}^\times$ is a homomorphism since $\psi_2(g) = \operatorname{tr} \pi_2(g)$ of degree $\psi_2(1) = 1$.

Its kernel is $\{g \in G : \psi_2(g) = 1\}$
$= \{$ elements conjugate to $()$, $(12)(34)$ or $(123)\}$
$= A_4$.

Another normal subgp: all elements $g \in G$ s.t. $\psi_3(g) = \psi_3(1) = 2$

<u>Theorem</u> Let $\pi : G \to GL_n(\mathbb{C})$ be a representation with character $\chi(g) = \operatorname{tr} \pi(g)$. (Its degree is $n = \chi(1)$.) The kernel of $\chi$ defined by

$$\ker \chi = \ker \pi = \{g \in G : \pi(g) = I\} = \{g \in G : \chi(g) = \chi(1) = n\}$$

is a normal subgroup of $G$.

Note: $\pi$ is a homomorphism; but $\chi$ is not a homomorphism unless $n=1$.

<u>Proof</u> If $g \in G$ then $\pi(g)$ is similar to $\begin{bmatrix} \varepsilon_1 & & 0 \\ & \varepsilon_2 & \\ 0 & & \ddots \; \varepsilon_n \end{bmatrix}$, where $\boxed{\varepsilon_i^m} = 1$ for all $i = 1, 2, \dots, n$.

$$\begin{bmatrix} \lambda & \alpha \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1}\alpha \\ 0 & \lambda^k \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

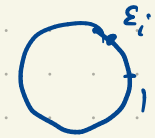
$\pi(g)$ is a root of $x^m - 1$

$m$ is the <u>exponent</u> of $G$ i.e. $g^m = 1$ for every $g \in G$

$\pi(g) \in GL_n(\mathbb{C})$ has order dividing $m$.

$\Rightarrow \chi(g) = \operatorname{tr} \pi(g) = \varepsilon_1 + \cdots + \varepsilon_n = n \iff \varepsilon_1 = \cdots = \varepsilon_n = 1. \iff \pi(g) = I$

$|\varepsilon_i| = 1 \Rightarrow \operatorname{Re} \varepsilon_i \leq 1$; equality iff $\varepsilon_i = 1$.




For $G = S_4$, there are four normal subgroups and they all arise as $\ker \chi$ for some $\chi$.

$\ker \chi_5 = \{()\}$

$\ker \chi_3 = \{(), (12)(34), (13)(24), (14)(23)\}$

$\ker \chi_2 = A_4$

$\ker \chi_1 = S_4$.

Recall: Let $H \leq G$. Then $H$ is normal in $G$ ($H \trianglelefteq G$) iff $H$ is a union of conjugacy classes, iff $H$ is an intersection of kernels of irreducible characters.

In this way we "read off" all the normal subgroups of $G$ from the char. table.

eg. $G = \{\pm 1\} \times \{\pm 1\} = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ Klein

Char table

| $\lvert C_G(g)\rvert$ | 4 | 4 | 4 | 4 |
|---|---|---|---|---|
| $g$ | $(1,1)$ | $(1,-1)$ | $(-1,1)$ | $(-1,-1)$ |
| $\chi_1$ | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | 1 | $-1$ | $-1$ |
| $\chi_3$ | 1 | $-1$ | 1 | $-1$ |
| $\chi_4$ | 1 | $-1$ | $-1$ | 1 |

Application to Frobenius groups.

Suppose $G \leq S_n$ ($G$ permutes $\{1, 2, \cdots, n\}$). More generally, $G$ permutes any set $X$, $G \leq \mathrm{Sym}(X) = \{\text{permutations of } X\}$ $X$ : any set. e.g. $X = \{1, 2, \cdots, n\}$

$G$ is <u>transitive</u> if for all $x, y \in X$, there exists $g \in G$ mapping $x \longmapsto y$.

The <u>stabilizer</u> of a point $x \in X$ is $G_x = \{g \in G : g(x) = x\}$.

Of course $G_x \leq G$.

The <u>orbit</u> of $x \in X$ is $x^G = \{g(x) : g \in G\} \subseteq X$.
or $G(x)$

$|x^G| = [G : G_x] = \frac{|G|}{|G_x|}$ (orbit-stabilizer formula).

$x^G = X$ iff $G$ is transitive.

$G$ is a <u>Frobenius group</u> if

(i) $G$ is transitive

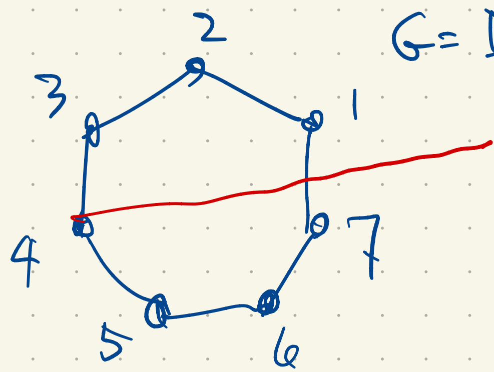
(ii) Point stabilizer is nontrivial $\left( |G_x| = \frac{|G|}{|X|} > 1 \right)$

(iii) The stabilizer of any two points is trivial i.e. $G_x \cap G_y = 1$ for all $x \neq y$ in $X$.

If $m$ is an odd positive integer then the symmetry group of a regular $m$-gon (i.e. the dihedral group of order $2m$) is a subgroup of $S_m$ having

$m$ rotations,
$m$ reflections.

eg. $m=7$



$G = D_7 = \langle (1234567), (17)(26)(35) \rangle$

$G_4 = \langle (17)(26)(35) \rangle$

$()$ = identity in $G$ fixes 7 points;
nontrivial rotations fix 0 " ;
other elements fix 1 point.

This is a Frobenius group.

There are exactly two groups of order 21: the cyclic group, and a Frobenius group $\langle (1234567), (124)(365) \rangle < S_7$ transitive.

$\downarrow$ conjugate by $(124)(365)$

$(2461357) = (1357246) = (1234567)^2$.

Eg. $G$ = direct isometries of $\mathbb{R}^2$ = {translations} $\cup$ {rotations}
(orientation-preserving)
is a Frobenius group. There are lots of finite analogues of this example.

e.g. $F = \mathbb{F}_{11} = \{$integers mod 11$\}$.

The affine general linear group on $F^2$ is a subgroup of $S_{121}$ consisting of transformations $v \longmapsto Av+b$, $A \in GL_2(F)$, $b \in F^2$.

$GL_2(F) = \{$invertible linear transformations on $F^2\}$

$|GL_2(F)| = (11^2-1)(11^2-11) = 120 \cdot 110 = 13200$

$|AGL_2(F)| = 11^2 \cdot 13200$ transitive on $F^2$.

Stabilizer of $0 = \begin{bmatrix}0\\0\end{bmatrix} \in F^2$ is $GL_2(F)$ of order 13200.

This group $AGL_2(F)$ is not a Frobenius group e.g.

$v \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix} v + \begin{bmatrix}0\\0\end{bmatrix}$ fixes all eleven vectors $\begin{bmatrix}a\\0\end{bmatrix} \longmapsto \begin{bmatrix}1&1\\0&3\end{bmatrix}\begin{bmatrix}a\\0\end{bmatrix} = \begin{bmatrix}a\\0\end{bmatrix}$

The two distinct points $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$ are fixed by more than just identity

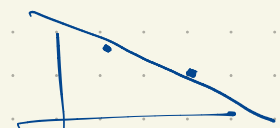
in fact the subgroup $\begin{bmatrix}1&b\\0&c\end{bmatrix}$ $b, c \in F$, $c \neq 0$ of order 110

Modify the example: $GL_2(F)$ has a subgroup isomorphic to $SL_2(\mathbb{F}_5)$ (order 120), sharply ~~transitive~~ transitive on the 120 nonzero vectors

The affine linear transformations on $F^2$ of the form $v \longmapsto Av+b$, $A \in$ subgp of $GL_2(\mathbb{F}_{11})$ isomorphic to $SL_2(\mathbb{F}_5)$, $b \in \mathbb{F}_{11}^2$ forms a Frobenius group of order $121 \cdot 120$. Actually this example is sharply 2-transitive

On $\mathbb{R}^2$, the transformations $\mathbb{R}^2 \to \mathbb{R}^2$ which are direct similarities is a Frob. gp., sharply 2-trans.

Let $F$ be any field. An affine linear map $F \to F$ is a map $x \mapsto mx+b$, $m \neq 0$. (invertible)



This group is $AGL_1(F)$.

Now if $H$ is a subgroup of $F^\times = \{\text{nonzero elements of } F\}$ then the affine linear transformations $F \to F$, $x \mapsto mx+b$, $x \in H$, $b \in F$ is a Frobenius group. (sharply 2-trans. iff $H = F^\times$).

$$AGL_1(F) \cong \left\{ \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} : m, b \in F,\ m \neq 0 \right\} < GL_2(F)$$

$$G = AGL_n(F) \cong \left\{ \left[\begin{array}{c|c} A & \begin{matrix} * \\ * \\ * \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] : A \in GL_n(F) \right\} < GL_{n+1}(F).$$

has subgroups
$$K = \left\{ \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] : A \in GL_n(F) \right\} \cong GL_n(F)$$

$$H = \left\{ \left[\begin{array}{c|c} I & \begin{matrix} * \\ * \\ * \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right] \right\} \cong \text{translations of } F^n, \quad x \mapsto x+b$$

$$H \trianglelefteq AGL_n(F)$$

$$K = G_0 = \text{stabilizer of } 0 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

This is 2-transitive on $F^n$, not sharply in general.

If we replace $GL_n(F)$ by a subgroup having no nontrivial elements with eigenvalue 1 we get a Frobenius group.

**Theorem** (Frobenius) Every Frobenius group $G$ has the form $G = K \rtimes H$ i.e. subgroups $H, K$ satisfy $G = KH$, $K \trianglelefteq G$, $K \cap H = 1$.

(Note: $KH = \{kh : h \in H, k \in K\}$ is a subgroup assuming one of them is normal.)

$H = G_0$ where $0 \in X$ any point in $X$
$= \{g \in G : g(0) = 0\}$

$$G/K = KH/K \cong H/K\cap H \cong H.$$

what is $K$? $K = \{1\} \cup \{$elements of $G$ which don't fix any point$\}$

$$= \{1\} \cup \left(G - \bigcup_{g \in G} (gHg^{-1})\right) \subseteq G.$$

Since $G$ is transitive, the stabilizer of any point $x \in X$ is conjugate to $H = G_0$. Why? Every point $x \in X$ has the form $x = g(0)$ for some $g \in G$.

If $h \in G_0 = H$ then $(ghg^{-1})(x) = (ghg^{-1})(g(0)) = gh(0) = g(0) = x$
$h(0) = 0$ $\qquad 0 = g^{-1}(x)$

i.e. $ghg^{-1} \in G_x$. So $gHg^{-1} \subseteq G_x$ i.e. $gG_0g^{-1} \subseteq G_x$ $\qquad G_0 \subseteq g^{-1}G_x g$

$G_0 \supseteq g^{-1}G_x g$

If $f \in G_x$ i.e. $f(x) = x$ then $(g^{-1}fg)(0) = g^{-1}f(x) = g^{-1}x = 0$.

The problem is: the subset $K$ defined above needs to be a subgp.
If I can show this, normality follows immediately.

In $S_5$ (left-to-right), $(1\,2\,3\,4\,5)(1\,5\,3\,4\,2) = (2\,4\,3)$

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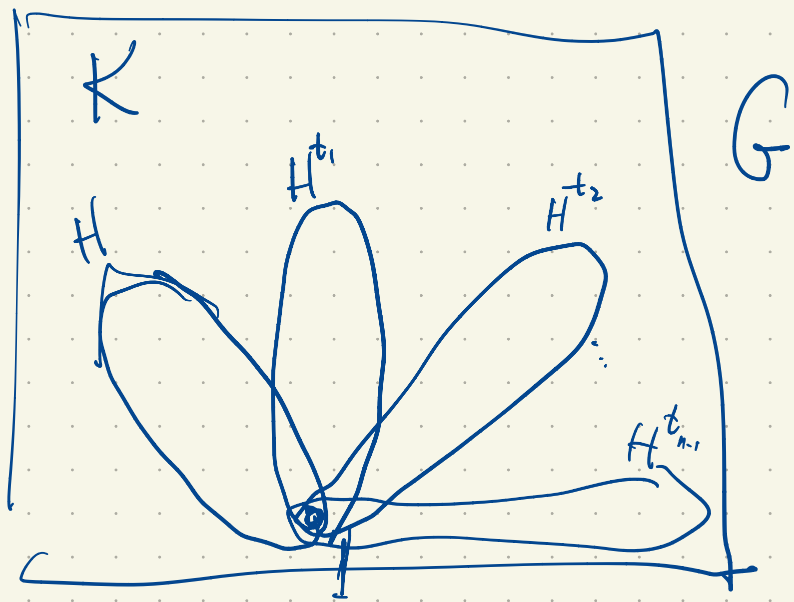
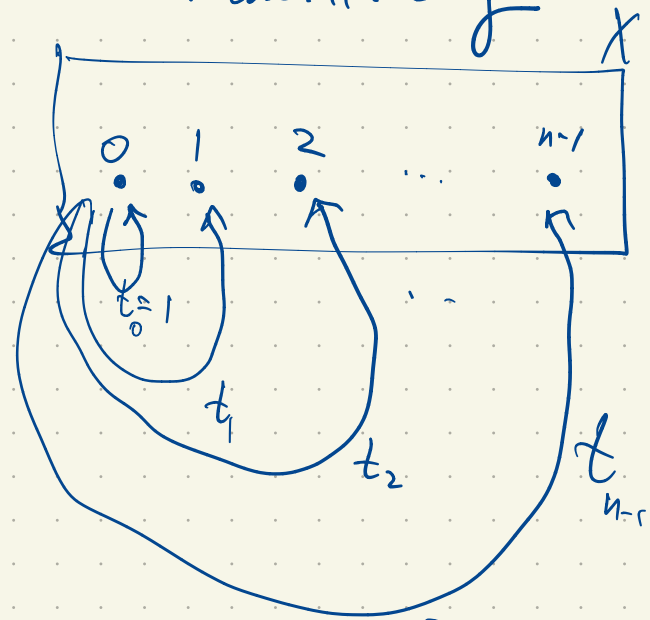



$H = G_0 = \{g \in G : g(0) = 0\}$

$0 \in X$

$G$ permutes $X$ transitively




Any two stabilizers $t_i^{-1} H t_i$, $t_j^{-1} H t_j$ are disjoint for $i \neq j$:

$H^{t_i} \cap H^{t_j} = 1$ for $i \neq j$

$G_i = \{g \in G : g(i) = i\} = \underbrace{t_i^{-1} H t_i}_{H^{t_i}}$

$T = \{t_0, t_1, \dots, t_{n-1}\}$ right transversal for $H$ in $G$

$$G = \bigsqcup_{i=0}^{n-1} H t_i$$

Every conjugate of $H$ has $|H| - 1 = \frac{|G|}{n} - 1$ non-identity elements.

$$K = \{1\} \cup \left(G \setminus \bigcup_{i=0}^{n-1} H^{t_i}\right)$$

$$|K| = 1 + \left(|G| - 1 - n\left(\frac{|G|}{n} - 1\right)\right)$$
$$= 1 + |G| - 1 - |G| + n = n.$$

$n|H| = |G|$

$|H| = \frac{|G|}{n}$

Why is $K$ a subgroup of $G$? Normality is easy.
We show that $K$ is an intersection of kernels of characters of $G$.

<u>Lemma</u> Let $\theta$ be a class function on $H$. Such that $\theta(1) = 0$. Then $\theta^G|_H = \theta$ i.e. $\theta^G(h) = \theta(h)$ for all $h \in H$.

<u>Proof</u> $\theta^G(g) = \sum_{i=0}^{n-1} \hat{\theta}(t_i^{-1} g t_i)$ where $\hat{\theta}(g) = \begin{cases} \theta(g), & \text{if } g \in H; \\ 0, & \text{if } g \notin H. \end{cases}$

For $g = 1$, $\theta^G(1) = \sum_i \hat{\theta}(1) = \sum_i \underbrace{\theta(1)}_{0} = 0 = \theta(1)$

$t_i^{-1} 1 t_i = 1$

For $h \in H$, $1 \neq h$,

$$\theta^G(h) = \sum_i \hat{\theta}(t_i^{-1} h t_i) = \hat{\theta}(h) = \theta(h)$$

as required. $\square$

When is $t_i^{-1} h t_i \in H$?
i.e. $t_i^{-1} h t_i \in H \cap t_i^{-1} H t_i$
Only if $i = 0$, $t_0 = 1$, $t_i^{-1} h t_i = h$

---

Now let $\psi$ be any irreducible character of $H$, $\psi \neq \underset{\text{principal character}}{1_H}$ $\quad 1_H(h) = 1$.

Define $\theta(h) = \psi(h) - \psi(1)$ i.e. $\theta = \psi - \psi(1) 1_H$.
So $\theta$ is a class function on $H$ with $\theta(1) = \psi(1) - \psi(1) = 0$.
This yields an induced class function $\theta^G$ on $G$ satisfying $\theta^G(h) = \theta(h) = \psi(h) - \psi(1)$ for all $h \in H$.

Define $\psi^* = \theta^G + \psi(1) 1_G$. This is a class function on $G$.

$$\psi^*(g) = \theta^G(g) + \psi(1)$$

We will prove that this is an irred. char. of $G$ and furthermore $\bigcap_{\psi \in \mathrm{Irr}_H} \ker \psi^* = K$

$\psi \in \mathrm{Irr}_H, \quad \psi \neq 1_H.$

Frobenius reciprocity: $[\theta^G, \rho]_G = [\theta, \rho|_H]_H$

$\theta$ class function on $H$

$\rho$ " " " $G$

$\theta = \psi - \psi(1) 1_H$

$$
\begin{aligned}
[\theta, \theta]_H &= [\psi - \psi(1) 1_H, \psi - \psi(1) 1_H]_H \\
&= [\psi, \psi]_H - [\psi, \psi(1) 1_H]_H - [\psi(1) 1_H, \psi]_H + [\psi(1) 1_H, \psi(1) 1_H]_H \\
&= 1 - \overline{\psi(1)} \underbrace{\cancel{[\psi, 1_H]_H}}_{0} - \psi(1) \underbrace{\cancel{[1_H, \psi]_H}}_{0} + \psi(1)^2 \underbrace{[1_H, 1_H]_H}_{1} \\
&= 1 + \psi(1)^2
\end{aligned}
$$

$$
\begin{aligned}
[\psi^*, \psi^*]_G &= [\theta^G + \psi(1) 1_G, \theta^G + \psi(1) 1_G]_G = [\theta^G, \theta^G]_G + \psi(1)[\theta^G, 1_G]_G + \psi(1)[1_G, \theta^G]_G + \psi(1)^2 \underbrace{[1_G, 1_G]_G}_{1} \\
&= \underbrace{[\theta, \theta^G|_H]_H}_{[\theta,\theta]_H = 1 + \psi(1)^2} + \psi(1) \underbrace{[\theta, 1_H]_H}_{} + \psi(1)[1_H, \theta]_H + \psi(1)^2
\end{aligned}
$$

$[\psi - \psi(1) 1_H, 1_H]_H$

$= \underbrace{\cancel{[\psi, 1_H]_H}}_{0} - \psi(1) \underbrace{[1_H, 1_H]_H}_{1}$

$-\psi(1)^2$

$-\psi(1)^2$

$= 1.$

$\psi^* = \sum a_i \chi_i, \quad a_i \in \mathbb{Z}, \quad \chi_i \in \mathrm{Irr}\, G.$

$= \theta^G + \psi(1) 1_G$

$\theta^G = \psi^G - \psi(1) 1_H^G$

$\psi^*(1) = \theta^G(1) + \psi(1) = \cancel{\theta(1)}^{0} + \psi(1) = \psi(1) \in \{1, 2, \cdots\}$

(a pos. integer)

$\Rightarrow \psi^* \in \mathrm{Irr}_G.$

$[\psi^*, \psi^*]_G = \sum a_i^2 = 1 \Rightarrow$ one of $a_i = \pm 1$ and all other terms are zero.

$\Rightarrow \psi^* = \pm$ (irred. char. of $G$)

Let $M = \bigcap_{\psi \in \mathrm{Irr}_H} \ker \psi^* \trianglelefteq G$.

$= \bigcap_{1_H \neq \psi \in \mathrm{Irr}_H} \ker \psi^*$

Claim: $M = K$

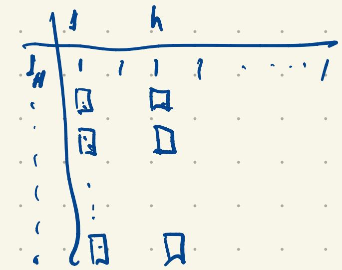
Prove containment in both directions

Suppose $h \in H \cap M$, $h \neq 1$. (get a contradiction?)

Then for every $\psi \in \mathrm{Irr}_H$, $1 \in \ker \psi^*$

$\Rightarrow \underset{\psi(h)}{\psi^*(h)} = \psi^*(1) = \psi(1) \Rightarrow h \in \ker \psi$.

Char table of $H$



$\psi^* = \theta^G + \psi(1) 1_G$

$\psi^*(h) = \underbrace{\theta^G(h)} + \psi(1)$

$\theta(h) = \psi(h) - \psi(1)$

$M$ contains no nonidentity element of $H$.

$\Rightarrow$ " " " " " ... $gHg^{-1}$, $g \in G$.

Contradicts orthog. of columns.

$\Rightarrow M \subseteq K$.

$|M| = |K| = n$.

Given a finite group $G$, where do irreducible representations of $G$ come from?
It turns out that they can all be found "inside" the group algebra $\mathbb{C}G$.
An algebra is (usually) is a ring which is also a vector space.
eg. $M(n,\mathbb{C}) = \{n\times n \text{ complex matrices}\}$ is an algebra of dimension $n^2$ over $\mathbb{C}$.

An algebra $A$ over a field $F$ has three basic binary operations:
Elements of $A$ can be thought of as "vectors"; elements of $F$ are "scalars".

in $F$: scalar + scalar = scalar
scalar × scalar = scalar

In $A$: vector + vector = vector
vector × vector = vector
scalar × vector = vector.

These give the ring structure for $A$.

These make $A$ into a vector space.

Axioms:
$(ab)v = a(bv)$ $\quad a,b\in F;\ v\in A$
$a(vw) = (av)w$ $\quad a\in F,\ v,w\in A$
$(uv)w = u(vw)$ $\quad u,v,w\in A$

In general, our algebras need not be commutative ie. as a ring i.e. $vw\neq wv$ in general $(v,w\in A)$.
Our algebras will have identity $1_A$, $1_A v = v$ $\quad 1_F = 1\in F$ scalar identity
Consider $M(n,\mathbb{C})$: $n\times n$ complex matrices. Noncommutative algebra with identity $I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$.

If $A_1, A_2, \dots, A_k$ are algebras over $\mathbb{C}$ then so is $\bigoplus_{i=1}^{k} A_i = A_1\oplus\cdots\oplus A_k$ with componentwise operations.

$M(n_1,\mathbb{C})\oplus M(n_2,\mathbb{C})\oplus\cdots\oplus M(n_k,\mathbb{C})$ is an algebra of dimension $\sum_{i=1}^{k} n_i^2 = n_1^2 + \cdots + n_k^2$

$$\cong \left\{\begin{bmatrix} A_1 & & & 0\\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_k\end{bmatrix} : A_i\in M(n_i,\mathbb{C})\right\} \subseteq M(n,\mathbb{C}),\quad n=\sum n_i$$
subalgebra

Group algebra: Let $G$ be a finite group.

The group algebra $\mathbb{C}G$ is the set of (formal) (symbolic) linear combinations of group elements

$$\mathbb{C}G = \left\{ \sum_{g \in G} a_g g \ : \ a_g \in \mathbb{C} \right\}$$

$$\mathbb{C}S_3 = \left\{ a_1() + a_2(12) + a_3(13) + a_4(23) + a_5(123) + a_6(132) \ : \ a_1, \dots, a_6 \in \mathbb{C} \right\}.$$

$$\big(5() + 2(12) - 7(13)\big)\big(4(12) + 5(13)\big) = 20(12) + 25(13) + 8() + 10(123) - 28(132) - 35()$$

$$= -27() + 20(12) + 25(13) + 10(123) - 28(132).$$

$\mathbb{C}G$ is an algebra of dimension $|G|$.

$$\mathbb{C}S_3 \cong M(1,\mathbb{C}) \oplus M(1,\mathbb{C}) \oplus M(2,\mathbb{C}) \cong \left\{ \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & d \\ 0 & 0 & e & f \end{bmatrix} \ : \ a,b,c,d,e,f \in \mathbb{C} \right\}$$

These are really the three irreducible repns of $S_3$.

$$\pi : S_3 \to M(4,\mathbb{C})$$

$$\pi(g) = \left[\begin{array}{c|c|cc} \pi_1(g) & 0 & 0 & 0 \\ \hline 0 & \pi_2(g) & 0 & 0 \\ \hline 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\pi_3(g)$}} \\ 0 & 0 & & \end{array}\right]$$

| Representation-theoretic terminology | Module terminology |
|---|---|
| vector space over $F$ with representation of $G$ | $FG$-module |
| invariant subspace | submodule |
| irreducible representation | simple module |
| completely reducible | semisimple |
| equivalent representations | isomorphic modules |
| $G$-equivariant linear transformation | module homomorphism |
| trivial representation $g \mapsto (1)$ | trivial $FG$-module $F$ |

TABLE 2.2: Glossary

Let $A$ be an algebra over a field $F$. (ring + vector space over $F$)
(think of examples: $M(n, \mathbb{C})$, $\mathbb{C}G$ )

Let $M$ be an additive abelian group (usually a vector space over $\mathbb{C}$).
We say $A$ acts on $M$ ($M$ is a (left) module for $A$) if we have binary operation

$$A \times M \longrightarrow M \qquad (a, m) \longmapsto am$$

such that
$$(a+a')m = am + a'm$$
$$a(m+m') = am + am'$$
$$(aa')m = a(a'm)$$
$$1_A m = m$$

$a, a' \in A$; $m, m' \in M$

$1_A \in A$ identity

Examples: $A = F$ a field.
$M$ is a module over $F$ iff $M$ is a vector space over $F$.

Eg. $A$ any algebra.
Then $A$ can act on itself ($M = A$). This is the regular $A$-module

A module over $A$ is an $A$-module.

Eg. $A = M(n, \mathbb{C})$ algebra acting on $M = \mathbb{C}^n$. So avoid thinking of elements of $M$ as being bigger than elements of $A$.

Eg. $\pi : G \longrightarrow GL(V) = \{$ invertible linear transformations $V \to V \}$ homomorphism ie. representation of $G$.

$$GL(\mathbb{C}^n) = GL_n(\mathbb{C})$$

For us, $G$ is a finite group.

gives rise to a $\mathbb{C}G$-module $V$. We have an action of the entire group algebra $\mathbb{C}G$ on $V$.

If $\alpha = \sum_{g \in G} a_g g \in \mathbb{C}G$ then we extend $\pi$ to a representation of $A = \mathbb{C}G$

ie. a homomorphism of algebras $\pi : \mathbb{C}G \longrightarrow M(n, \mathbb{C})$

$$\pi(\alpha) = \pi\Big(\sum_g a_g g\Big) = \sum_g a_g \pi(g) \in M(n, \mathbb{C}) \qquad (a_g \in \mathbb{C})$$

If $A$ and $B$ are algebras over $\mathbb{C}$ then a homomorphism $A \to B$ is a map $f: A \to B$ such that

$$\begin{cases} f(ca + c'a') = cf(a) + c'f(a') \in B \\ f(aa') = f(a)f(a') \end{cases} \quad \text{for all } c, c' \in \mathbb{C},\ a, a' \in A$$

$\mathrm{Hom}_{\mathbb{C}}(A, B) = \{$ algebra homomorphisms $A \to B \}$ is a vector space over $\mathbb{C}$

$\mathrm{End}_{\mathbb{C}}(A) = \mathrm{Hom}_{\mathbb{C}}(A, A) = \{\underbrace{\text{endo}}_{\text{into}}\text{morphisms } A \to A\}$ is an algebra over $\mathbb{C}$

$$\mathrm{End}_{\mathbb{C}}(\mathbb{C}^n) \cong M(n, \mathbb{C})$$

Suppose $M, M'$ are modules over $A$.

A map $f: M \to M'$ is $\mathbb{C}$-linear ($f \in \mathrm{Hom}_{\mathbb{C}}(M, M')$) if $f(cm + c'm') = cf(m) + c'f(m')$ for all $c, c' \in \mathbb{C}$, $m, m' \in M$.

$\mathrm{Hom}_{\mathbb{C}}(M, M')$

We say $f: M \to M'$ is $A$-linear ($f \in \mathrm{Hom}_A(M, M')$) if $f(am + a'm') = af(m) + a'f(m')$.

$A$-homomorphism $M \to M$

$f$ is $A$-linear $\Rightarrow$ $f$ is $\mathbb{C}$-linear.

---

Look at $A = \mathbb{C}G$ group algebra over $\mathbb{C}$.

Modules over $A = \mathbb{C}G$ are the same thing as vector spaces $\mathbb{C}^n$ over $\mathbb{C}$ having a group action i.e. $\pi: G \to GL_n(\mathbb{C})$.

For $v \in \mathbb{C}^n$, $g \in G$, $gv = \pi(g)v$

Let's consider $A = M(n, \mathbb{C})$ as a prelude to $\mathbb{C}G$.

Think of $A$ acting on itself by left-multiplication, acting on $M = M(n, \mathbb{C})$.

What are the $\begin{cases} \text{left ideals} \\ \text{right ideals} \\ \text{two-sided ideals} \end{cases}$ of $M(n, \mathbb{C})$?

$\left\{ \begin{bmatrix} * & \\ * & \\ \vdots & \bigcirc \\ * & \end{bmatrix} \right\}$ is a left ideal. (a submodule of the regular module)

$$\underset{k\times l}{B} \underset{l \times n}{[v_1, \cdots, v_n]} = \underset{k\times n}{[Bv_1, Bv_2, \cdots, Bv_n]} \qquad B[v_1, 0, \cdots, 0] = [Bv_1, 0, \cdots, 0]$$

column vectors

$V_1 = \begin{bmatrix} * & \\ * & \\ \vdots & \bigcirc \\ * & \end{bmatrix} \subseteq M(n, \mathbb{C})$ is a left ideal i.e. submodule of the regular $A$-module $(A = M(n, \mathbb{C}))$ isomorphic to the $A$-module $\mathbb{C}^n$.

$\mathbb{C}^n \cong_A V_1$ i.e. stronger that isomorphism as vector spaces over $\mathbb{C}$ $\quad V_1 \cong_{\mathbb{C}} \mathbb{C}^n$

$V_i \cong_A \mathbb{C}^n$ (iso. as $A$-modules)

$$v_1 \xmapsto{\phi} [v_1, 0, 0, \cdots, 0]$$

$\phi(cv_1 + c'v_1') = c\,\phi(v_1) + c'\phi(v_1')$ $\quad c, c' \in \mathbb{C}; \quad v_1, v_1' \in \mathbb{C}^n$ $\qquad \mathbb{C}$-linear

$\phi(Bv_1) = B\phi(v_1)$ for all $B \in A = M(n, \mathbb{C})$ i.e. $\phi$ is $A$-equivariant

$\phi$ preserves the action of $A$.

The regular $A$-module is a direct sum of simple $A$-modules

$A = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ where $V_1 = \begin{bmatrix} * & \\ * & \\ \vdots & \bigcirc \\ * & \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & * & \\ 0 & 0 & \\ \vdots & \vdots & \bigcirc \\ 0 & * & \end{bmatrix}$, $\cdots$, $V_n = \begin{bmatrix} & * \\ \bigcirc & * \\ & \vdots \\ & * \end{bmatrix}$

$\dim A = n^2$, $\quad \dim V_i = n$. $\quad V_i \subseteq A$ left ideal.

Each $V_i$ is a simple $A$-module.

$A = M(n, \mathbb{C})$ is transitive on the nonzero column vectors in $\mathbb{C}^n$.

If $v_0 \in \mathbb{C}^n$ is nonzero then $Av_0 = \mathbb{C}^n$.

The left ideal $V_i \subseteq A$ is minimal ideal i.e. simple module. i.e. irreducible representation of the algebra $A$.

$A$ has infinitely many minimal left ideals for $n > 1$

eg. $A = M(3, \mathbb{C})$ has minimal left ideals $V_1 = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 & * & 0 \\ 0 & * & 0 \\ 0 & * & 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{bmatrix}$,

$\left\{ \begin{bmatrix} a & a & 0 \\ b & b & 0 \\ c & c & 0 \end{bmatrix} : a, b, c \in \mathbb{C} \right\}$ $\qquad B[v_1, v_1, 0] = [Bv_1, Bv_1, 0]$

For every row vector $w = [w_1, w_2, \cdots, w_n]$, $w_i \in \mathbb{C}$

$V_w = \{ vw : v \in \mathbb{C}^n \}$ $\qquad B(vw) = (Bv)w \in V_w$

$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} [w_1\, w_2 \cdots w_n]$

There are many ways to decompose $A = \bigoplus_{i=1}^{n} V_i$ as a direct sum of $n$ minimal left ideals / simple left ideals.

But all the minimal left ideals of the regular module are isomorphic to $\mathbb{C}^n$ as $A$-modules.

The only 2-sided ideals of $A$ are $0$ and $A$.

$A$ is a simple algebra.

Let $M$ be an (left) $A$-module. $M$ is simple if its only submodules are $0$ and $M$.

eg. $A = M(n, \mathbb{C})$, $\mathbb{C}^n$ is a simple module.

$M$ is semisimple if $M$ is a direct sum of simple modules.

$\Longleftrightarrow$ every submodule of $M$ has a complementary submodule. (p4)

$\boxed{\text{Stick to finite dimensional}}$

For $G$ any finite group, $A = \mathbb{C}G$ group algebra, every (finite dimensional) module for $A$ (module/representation for $G$) is semisimple.

This is Maschke's theorem, which we have proved. (p.5-6 of handout)

Every irreducible representation of $G$ is isomorphic to a minimal left ideal of $A = \mathbb{C}G$

<u>Theorem</u> (p.6) Let $R$ be a semisimple algebra. Then Let $M$ be a simple (minimal: no nonzero submodules) $R$-module (left module). Then $M$ is isomorphic to a minimal left ideal $I \subseteq R$.

<u>Proof</u> Let $v \in M$ be nonzero. Then $Rv \subseteq M$ is a nonzero submodule. $(R \cdot Rv \subseteq Rv)$

Since $M$ is simple, $Rv = M$. Let $\phi : R \to M$, $r \mapsto rv$. This is a homomorphism of $R$-modules. (If $s \in R$ then $\phi(sr) = srv = s(rv) = s\phi(r)$.)

The annihilator of $v$ in $R$ is $\ker\phi = \{r \in R : rv = 0\} \subseteq R$ is an ideal since it's the kernel of a homomorphism, (so $\ker\phi$ is a submodule of the regular module).

So $R = \ker\phi \oplus J$ for some ideal $J \subseteq R$. The First Isomorphism theorem yields $M = \phi(R) = Rv = R/\ker\phi \cong J$. $\square$.

In the case $R = \mathbb{C}G$, $|G| < \infty$, $\pi : G \to GL_n(\mathbb{C})$ irreducible representation, then $M = \mathbb{C}^n$ is a module for $R$. $\Big(\underbrace{\sum_{g\in G} a_g g}_{\in R}\Big) v = \sum_{g \in G} a_g \pi(g) v$.

We now know that $\pi$ comes from some minimal left ideal of $R = \mathbb{C}G$.

Why are there only finitely many irred. repns. of $G$ up to equivalence?

There are infinitely many left ideals of $\mathbb{C}G$ in general, <u>but</u> only finitely many up to isomorphism.

**Theorem** (p.7) Let $R$ be semisimple, so $R = I_1 \oplus I_2 \oplus \cdots \oplus I_m$ where each $I_i$ is a minimal left ideal of $R$. Then every simple $R$-module is isomorphic (as an $R$-module) to $I_i$ for some $i \in \{1, 2, \ldots, m\}$. In particular, $R$ has only finitely many simple modules up to isomorphism.
(Every finite group has only a finite number of irred. reprs. up to equivalence)

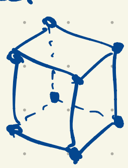
**Proof** Let $M$ be a simple $R$-module. By the previous result, WLOG $M$ is a minimal left ideal of $R$. So $R = M \oplus J$ for some left ideal $J$.
$J$ is a proper ideal ($J \subset R$, $J \neq R$) so there exists $i \in \{1, 2, \ldots, m\}$ such that $I_i \not\subseteq J$. Claim: $M \cong I_i$ as $R$-modules.

$$I_i \cong I_i / \underbrace{(I_i \cap J)}_{\substack{\text{proper sub-ideal of } I_i \\ \text{hence } I_i \cap J = 0}} \underbrace{\cong}_{\substack{\text{Second Isomorphism} \\ \text{Theorem}}} (J + I_i)/J \subseteq R/J \cong M.$$

must be equality since $M$ is minimal. $\square$

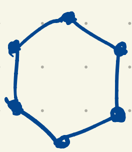
Let's take another detour: application to spectra of Cayley graphs.
L. Babai

Eg.



$G = \mathbb{F}_2^3$ under addition
$S = \{(100), (010), (001)\}$

$$\begin{bmatrix} 0&1&1&0&1&0&0&0 \\ 1&0&0&1&0&1&0&0 \\ 1&0&0&1&0&0&1&0 \\ 0&1&1&0&0&0&0&1 \\ 1&0&0&0&0&1&1&0 \\ 0&1&0&0&1&0&0&1 \\ 0&0&1&0&1&0&0&1 \\ 0&0&0&1&0&1&1&0 \end{bmatrix}$$

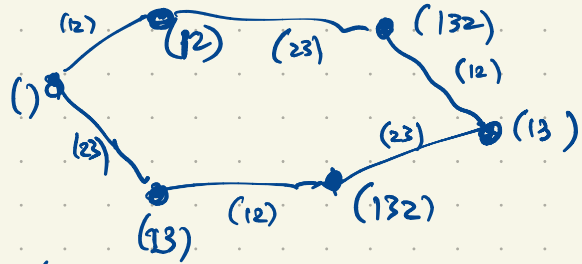


$G = S_3$
$S = \{(12), (23)\}$

$$\begin{bmatrix} 0&1&0&0&0&1 \\ 1&0&1&0&0&0 \\ 0&1&0&1&0&0 \\ 0&0&1&0&1&0 \\ 0&0&0&1&0&1 \\ 1&0&0&0&1&0 \end{bmatrix}$$

Let $G$ be a group and $S \subseteq G$ a generating set (i.e. $\langle S \rangle = G$) such that $1 \notin S$; $s \in S \Leftrightarrow s^{-1} \in S$.
The Cayley graph $\mathrm{Cay}(G, S)$ has vertex set $G$.
There is an edge between $g$ and $h$ ($g, h \in G$) iff there exists $s \in S$ such that $g = hs$
i.e. $h^{-1}g \in S$.
$\mathrm{Cay}(G, S)$ is a simple graph (undirected graph with no loops or multiple edges).

$\mathrm{Cay}(S_3, \{(12), (23)\}) =$




$\mathrm{Cay}(G,S)$ is a regular graph of degree $|S|$ with $|G|$ vertices.

It is vertex-transitive since $G$ acts on $\mathrm{Cay}(G,S)$ by left-multiplication

$$x \sim y \iff y = xs \iff gy = gxs \iff gx \sim gy$$

$x, y \in G$     for some $s \in G$

But transitive graphs don't have to be Cayley graphs.

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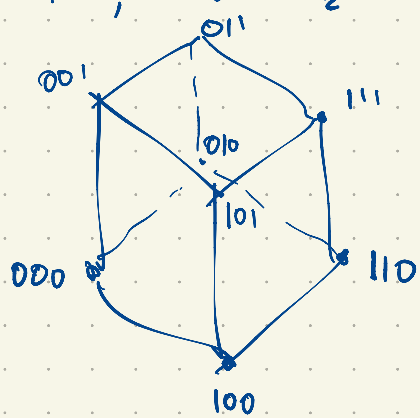
$G = F^3$, $\quad F = \mathbb{F}_2 = \{0,1\}$, $\quad S = \{(100), (010), (001)\}$,




$G$ has 8 irreducible characters, all linear (i.e. degree 1) indexed by $v \in G$.

$$\chi_v(u) = (-1)^{u \cdot v}$$

Given an abelian group $G$, $|G| = n$, there are $n$ irred. characters.

$S \subset G$ generators as above

The eigenvalues of the adjacency matrix $\lambda_1, \dots, \lambda_n$ are

$$\lambda_i = \sum_{s \in S} \chi_i(s) \qquad \mathrm{Irr}\, G = \{\chi_1, \chi_2, \dots, \chi_n\}, \quad \chi_i(1) = 1.$$

For the Hamming cube on 8 vertices, as above,

$$\lambda_v = \chi_v((100)) + \chi_v((010)) + \chi_v((001)) = (-1)^{v\cdot(100)} + (-1)^{v\cdot(010)} + (-1)^{v\cdot(001)}$$

$v \in \mathbb{F}_2^3$

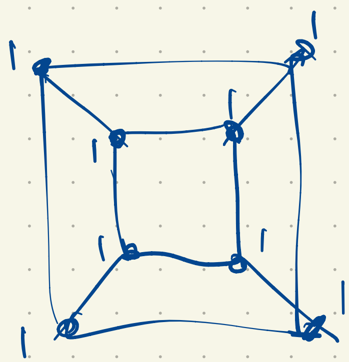
$$\lambda_{000} = (-1)^0 + (-1)^0 + (-1)^0 = 1+1+1 = 3$$

$$\lambda_{100} = (-1)^1 + (-1)^0 + (-1)^0 = -1+1+1 = 1 = \lambda_{010} = \lambda_{001}$$

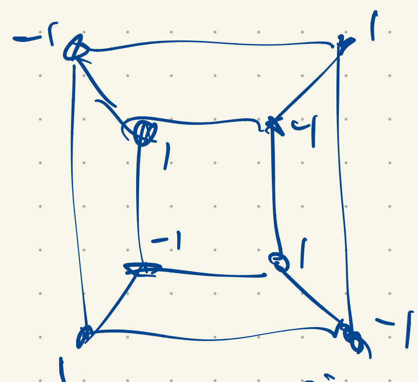
$$\lambda_{110} = (-1)^1 + (-1)^1 + (-1)^0 = -1 + (-1) + 1 = -1 = \lambda_{101} = \lambda_{011}$$

$$\lambda_{111} = -1 - 1 - 1 = -3$$

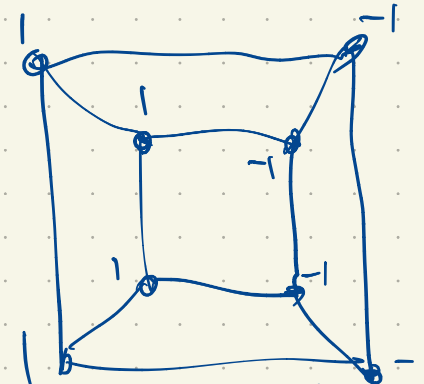
$\mathbb{C}G \xrightarrow{\text{multiply by } \sigma = \sum_{s \in S} s} \mathbb{C}S$



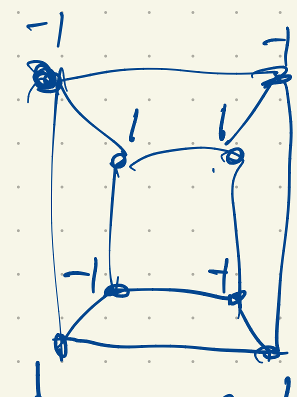
eigenvector with eigenvalue 3
multiplicity 1



eigenvalue $-3$.



eigenvalue 1 (multiplicity 3)



eigenvalue $-1$ (multiplicity 3)