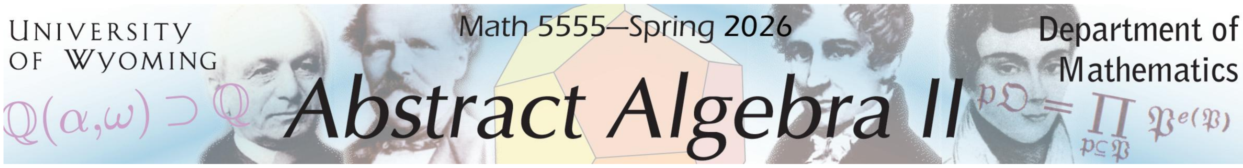


Homework Problems

(Due Wednesday, May 13, 2026 by 10:00am on WyoCourses)

Instructions: See the syllabus for general expectations regarding homework. In particular, remember that you are encouraged to work together but that anything you submit must be your own work. Work on any subset of these questions that interest you or that you feel comfortable with; you should *not* expect to answer all questions. I suggest that you submit solutions to individual problems as separate documents. (No limit has been set for the number of document submissions in WyoCourses.) Get an early start on your favorite problem, solve it and submit it, then come back to other problems as time permits. Don't wait until May 13. Please check this homework assignment repeatedly online as I may to add additional problems, corrections, and further hints and comments.

We begin with a subject strictly outside of representation theory, because it is very useful for some needed computations. (This topic belongs to Invariant Theory; see Appendix A6 from my notes on Incidence Geometry, available on my website.)

Consider the ring $R = \mathbb{C}[x_1, \ldots, x_n]$ of all polynomials in n indeterminates $x_1, \ldots, x_n$. A polynomial $f(x_1, \ldots, x_n) \in R$ is *symmetric* if $f(x_{\sigma(1)}, \ldots, f_{\sigma(n)}) = f(x_1, \ldots, x_n)$ for all $\sigma \in S_n$. Of the *many* examples, the only ones we shall consider here are the *power sums* (i.e. *moments*)

$$p_k = p_k(x_1, \ldots, x_k) = x_1^k + \cdots + x_n^k \quad \text{for } k = 0, 1, 2, \ldots,$$

and the *elementary symmetric polynomials*

$$\begin{aligned}
e_0 &= e_0(x_1, \ldots, x_n) = 1, \\
e_1 &= e_1(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \\
e_2 &= e_2(x_1, \ldots, x_n) = x_1x_2 + x_1x_3 + \cdots + x_{n-1}x_n, \\
&\cdots \\
e_k &= e_k(x_1, \ldots, x_n) = \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} x_{i_1} x_{i_2} \cdots x_{i_k}, \\
&\cdots \\
e_n &= e_n(x_1, \ldots, x_n) = x_1 x_2 \cdots x_n.
\end{aligned}$$

Note that e_k has $\binom{n}{k}$ terms; and $e_k = 0$ unless $0 \leqslant k \leqslant n$. Also,

$$(t + x_1)(t + x_2) \cdots (t + x_n) = \sum_{i=0}^{n} e_i t^{n-i},$$

so the product on the left may be viewed as a generating function for the elementary symmetric polynomials. This says that the coefficients of a monic polynomial in t are (up to a sign $(-1)^i$) the elementary symmetric polynomials in the roots of the polynomial.

The set of all symmetric polynomials in n indeterminates is a subalgebra of R. It is in fact the subalgebra generated by $e_1, \ldots, e_n$, or by $p_1, \ldots, p_n$. We of course mean 'generated as a subalgebra', i.e. via products and linear combinations. In other words, the elementary symmetric polynomials form the subalgebra

$$\mathbb{C}[e_1, e_2, \ldots, e_n] = \mathbb{C}[p_1, p_2, \ldots, p_n].$$

Given a symmetric polynomial $f(x_1, \ldots, x_n)$, we are often called upon to rewrite it as a polynomial in the e_i's, or as a polynomial in the p_i's. In such cases it is useful to know the algebraic relations between the two sets of generators. For example when $n = 3$, these relations are

$$\begin{aligned} p_1 &= e_1; & e_1 &= p_1; \\ p_2 &= e_1^2 - 2e_2; & e_2 &= \tfrac{1}{2}p_1^2 - \tfrac{1}{2}p_2; \\ p_3 &= e_1^3 - 3e_1e_2 + 3e_3; & e_3 &= \tfrac{1}{6}p_1^3 - \tfrac{1}{2}p_1p_2 + \tfrac{1}{3}p_3. \end{aligned}$$

Rather than try to write down *explicit* formulas for each set of generators in terms of the other, it is easier to use the Newton relations

$$ke_k = \sum_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i \qquad \text{for } 1 \leqslant k \leqslant n,$$

which one can use to *recursively* compute either list of generators in terms of the other.

1. **Newton relations for symmetric polynomials.** I have a 5×5 integer matrix B such that B^k has trace 3, 15, 27, 95, 243 for $k = 1, 2, 3, 4, 5$.
   (a) Determine the characteristic polynomial $f(x) = \det(xI - B)$ *using Newton's relations.*
   (b) Factor $f(x)$ into irreducible factors in $\mathbb{Z}[x]$ of degree at most 2.
   (c) Verify that B has five distinct real eigenvalues.

   Hint. The traces of the powers of B are in fact power sums of the eigenvalues. Using the Newton relations, we compute the elementary symmetric polynomials in the eigenvalues. By carefully considering the signs, determine the coefficients of $f(x)$. This trick for computing characteristic polynomials of a matrix is very handy since it is easy to implement using just a few lines of code. I hope you will find it useful on occasion, as I have.

Let G be a finite group of order n, and let $S \subset G$ be a subset such that $1 \notin S$; S is symmetric in the sense that $g \in S$ iff $g^{-1} \in S$; and $\langle S \rangle = G$. Then we define the *Cayley graph* $\text{Cay}(G, S)$ as the graph Γ having vertex set G; and two vertices $x, y \in G$ are *adjacent* (denoted $x \sim y$) iff $x^{-1}y \in S$. Our conditions on S mean that Γ is an ordinary graph (an undirected graph with no loops or multiple edges), although the theory of Cayley graphs works just as well with directed graphs, loops and multiple edges, with appropriate modifications. The graph Γ is vertex-transitive: right-multiplication by elements of G gives an automorphism group which is transitive on the vertices. (If the condition for adjacency is replaced by $xy^{-1} \in S$, then G we obtain a transitive group of automorphisms using instead the multiplication by elements of G on the left; but this graph is isomorphic to the one we have defined, using the bijection $g \mapsto g^{-1}$ on the vertices.) The *adjacency matrix* of Γ is the $n \times n$ matrix A whose rows and columns are indexed by the elements of G, having (x, y)-entry 0 or 1 according as $x \not\sim y$ or $x \sim y$ (i.e. $x^{-1}y \notin S$ or $x^{-1}y \in S$). The *spectrum* of Γ is the spectrum of A (i.e. the multiset of eigenvalues of A).

To compute the spectrum, let $\chi_1, \ldots, \chi_k$ be the irreducible characters of G. If G is abelian, then $k = n$ and the spectrum consists of the eigenvalues λ_i are given by

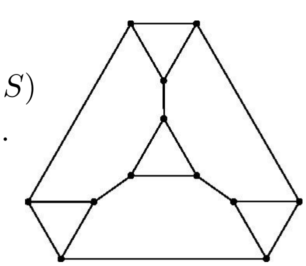
$$\lambda_i = \sum_{s \in S} \chi_i(s) \quad \text{for } i \in \{1, 2, \ldots, n\}.$$

In the general case, the degrees $n_i = \deg \chi_i = \chi_i(1)$ satisfy $\sum_{i=1}^{k} n_i^2 = n$. Here the eigenvalues are denoted $\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,n_i}$ for each $i \in \{1, 2, \ldots, k\}$; and each $\lambda_{i,j}$ is repeated times (giving the correct total number of eigenvalues $\sum_i n_i^2 = n$). For each i, the n_i distinct eigenvalues $\lambda_{i,j}$ are found as the solutions of a system of n_i equations in n_i unknowns:

$$\lambda_{i,1}^t + \lambda_{i,2}^t + \cdots + \lambda_{i,n_i}^t \;=\; \sum_{s_1, \ldots, s_t \in S} \chi_i(s_1 s_2 \cdots s_t) \qquad \text{for } t \in \{1, 2, \ldots, n_i\}.$$

Note that the sum on the right has $|S|^t$ terms. This method (Babai, 1979) was outlined in class, with two examples, one abelian and another nonabelian. The preferred method of solving for the eigenvalues $\lambda_{i,j}$, however, was only hinted at in class, as this uses the Newton relations explained above. Of course our character sums yield power sums of the eigenvalues $\lambda_{i,j}$, from which we obtain the elementary symmetric polynomials using Newton's relations. This gives a polynomial of degree n_i whose roots are $\lambda_{i,j}$ for $j = 1, 2, \ldots, n_i$.

2. **Spectrum of Cayley graph.** The Cayley graph $\Gamma = \mathrm{Cay}(A_4, S)$ with $S = \{(12)(34), (123), (132)\} \subset A_4$ is shown on the right. Determine the spectrum of this graph using the method described above.



In class we used character tables to show that the Monster has no subgroup isomorphic to the Baby Monster. (The Baby Monster is however involved in the Monster M, as a quotient group $C_M(\tau)/\langle\tau\rangle$ where $\tau \in M$ is an involution of type 2A.) Unfortunately I left out a final step in our proof. The following easy exercise closes the gap.

Recall (Feb 18) that we start by assuming that M has a subgroup isomorphic to B. From this we need to derive a contradiction. Let $\chi = \chi_2$ be the irreducible character of M of degree 196883, arising from the smallest nonprincipal representation of the Monster; and let ψ_1, ψ_2, ψ_3 be the three irreducible characters of B of smallest degree 1, 4371, 96255 respectively. The restriction $\chi|_B$ is a character of B, which must decompose as

$$(*) \qquad \chi|_B = k_1\psi_1 + k_2\psi_2 + k_3\psi_3 \text{ for some non-negative integers } k_1, k_2, k_3.$$

There is a unique conjugacy class of elements of order 11 (denoted 11A) in both B and M; so elements of class 11A in B, must also be of class 11A in M. The same is true for classes 1A, 17A, 19A, 25A, 38A and 55A in both groups. Evaluating both sides of $(*)$ at representatives of classes 1A, 11A and 17A gives three linear equations in the three unknowns k_1, k_2, k_3. These equations have a unique solution. In class (Feb 18) I wrote down the three equations explicitly, and I incorrectly stated that their solutions were nonintegral rational numbers, yielding the desired contradiction. Evidently I made a mistake in my arithmetic!

3. **The Baby Monster is not a subgroup of the Monster.** Determine the values of k_1, k_2, k_3, uniquely determined as explained above. These are in fact non-negative integers. To obtain the final contradiction, evaluate both sides of $(*)$ at a carefully chosen involution (element of order 2) in B.

Hint. There are four conjugacy classes of involutions in B, designated 2A, 2B, 2C, and 2D. You will need to indicate which conjugacy class of involution you are considering. Not all of them give the desired contradiction! Also there are two conjugacy classes of involutions in M, denoted 2A and 2B in that group, but we do not know how these correspond to classes in B. Nevertheless by choosing the right conjugacy class of involution in B, the final contradiction is easily obtained.

We remark that after incorporating the coefficients k_1, k_2, k_3 into $(*)$, this identity is satisfied for elements of classes 1A, 11A, 17A, 19A, 25A, 38A, and 55A. This seems quite remarkable. In class I asserted that the resulting linear equations were inconsistent over $\mathbb{Q}$, but evidently I was mistaken in this. We need to consider additional conjugacy classes to obtain the final contradiction. Fortunately, all the character values you need are clearly seen in the pdf slide from Feb 18.

In class we have computed the character tables of several small groups, including cyclic groups, S_3, both nonabelian groups of order 8, A_4, S_4, and A_5. The following exercises are not original and the answers can be found using many different sources. Obviously I am asking you to actually compute them yourself, using methods similar to those we used in class.

4. **An extraspecial group of order 27.** Let $G < GL_3(\mathbb{F}_3)$ be the multiplicative group of order 27 consisting of all 3×3 matrices over $\mathbb{F}_3$ having 1's on the main diagonal and zeroes below the main diagonal. This celebrated group is nonabelian, yet it has 26 elements of order 3, just like the elementary abelian group of order 27 (the additive group of a 3-dimensional vector space over $\mathbb{F}_3$). Determine the character table of G.

5. **A group of order 48.** Consider the subgroup $G < GL_3(\mathbb{R})$ of order 48 consisting of all monomial matrices having three nonzero entries ± 1, one in each row and column; and the remaining six entries are zero. Determine the character table of G.

6. **The simple group of order 168.** Determine the character table of $GL_3(\mathbb{F}_2)$, the simple group of order 168.

7. **The simple group of order 360.** Determine the character table of A_6, the simple group of order 360.