Solution to the 2-Hats Problem

The two participants confer and agree that one of them will be participant A and the other participant B. The strategy is for participant A to guess the same colour as the cap of participant B; and for participant B to guess the colour opposite to that of the cap of participant A. It is not hard to check (in each of the four cases, according to the colours of the hats as I assign them) that in every case, one participant will guess correctly and the other incorrectly. This guarantees their success.

Solution to the 7-Hats Problem

The simplest strategy (which is not optimal) succeeds with probability $\frac{1}{2} = 50\%$. To achieve this, one of the participants is designated to guess a colour (perhaps red, perhaps blue, and this guess can be chosen in advance or chosen randomly when the time comes). All other participants pass.

The optimal strategy succeeds with probability 87.5%. To achieve this, the seven participants confer and number off as participant 1, participant 2, …, participant 7. Using red=0 and blue=1, every assignment of hat colours to the participant's heads is then one of the 128 bitstrings of length 7. The participants must also agree on a binary Hamming code of length 7, i.e. a set of 16 bitstrings of length 7 which are at distance at least 3 apart from each other. Each participant, after seeing the other six caps, should translate what they see into a bitstring of length 7 with one missing entry '*'. For example participant 3 might see 10*0010 (meaning participants 1 and 6 have blue caps and the others have red caps). Each participant should guess as follows.

- If what the participant sees does not match any of the Hamming codewords, he/she should pass.
- If what the participant sees matches a Hamming codeword, he/she should guess the colour *opposite* to what the Hamming codeword

indicates in that position. For example if participant 3 sees 10*0010 and they have agreed upon a Hamming code which contains the codeword 1010010, then participant 3 should guess red (i.e. 0) as the colour of his/her cap.

If my assignment of hats turns out to be in the Hamming code that the participants have chosen, then all seven participants will guess incorrectly. But this will only happen $\frac{1}{8} = 12.5\%$ of the time.

Otherwise ($\frac{7}{8} = 87.5\%$ of the time) my assignment of hats turns out to not be in their Hamming code. In this case it has distance 1 from a unique Hamming codeword, and one participant will then correctly guess the correct assignment of hats, and the other six participants will pass.