
Fields
Book III

We have been talking about number fields: finite extensions $E \supseteq \mathbb{Q}$ i.e. $[E:\mathbb{Q}] = n < \infty$.
(Some are Galois i.e. $G = \mathrm{Aut}\, E$ satisfies $|G| = n$; but in general $|G| \leq n$.)

Back to basics:
In a field F, if $\underbrace{1+1+1+\cdots+1}_{n \geq 1} = 0$ then the smallest n for which this occurs is the characteristic of F.

If F has characteristic $n > 0$ then n must be prime. If $n = ab$, $a, b \geq 1$ then

$$\underbrace{(1+1+\cdots+1)}_{a}\underbrace{(1+1+\cdots+1)}_{b} = \underbrace{1+1+1+\cdots\cdots+1}_{n=ab} = 0$$

By minimality of n, n is prime.
If $\underbrace{1+1+\cdots+1}_{n} \neq 0$ for any $n \geq 1$, then we say n has characteristic 0.

Given a field F, $\mathrm{char}\, F$ = characteristic of F is either 0 or p (some prime p).

- If $\mathrm{char}\, F = p$ then $F \supseteq \mathbb{F}_p$ = field of order p ($\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, 2, \cdots, p-1\}$ = "integers mod p").
  eg. $\mathbb{F}_p$, $\mathbb{F}_{p^2}$, $\mathbb{F}_{p^3}$, $\mathbb{F}_{p^4}$, $\cdots$, $\mathbb{F}_p(x) = \{$all rational functions in x with coefficients in $\mathbb{F}_p\}$, $\cdots$

- If $\mathrm{char}\, F = 0$ then $F \supseteq \mathbb{Q}$. Eg. $\mathbb{R}, \mathbb{C}, \mathbb{Q}$, number fields, $\mathbb{A} = \{$algebraic numbers$\} \subset \mathbb{C}$
  eg. $\mathbb{Q}[\sqrt{2}]$

In either case F has a unique smallest subfield, either $\mathbb{F}_p$ or $\mathbb{Q}$, called the prime subfield of F.

All fields of characteristic 0 are infinite. (They are extensions of $\mathbb{Q}$, hence vector spaces over $\mathbb{Q}$.)

If $E \geq F$ is a field extension (i.e. E, F are fields with F a subfield of E) then E is a vector space over F. The dimension of this vector space is the degree $[E:F]$ of this extension eg.

$[\mathbb{C}:\mathbb{R}] = 2$ $\{1, i\}$ basis

$[\mathbb{R}:\mathbb{Q}] = \infty$ $1, \sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{10}, \sqrt{11}, \ldots$ are lin. indep.

$[\mathbb{C}:\mathbb{Q}] = \underbrace{[\mathbb{C}:\mathbb{R}]}_{2}\underbrace{[\mathbb{R}:\mathbb{Q}]}_{\infty} = \infty$

For fields of characteristic a prime p, some are finite, some are infinite.
Given p prime and $k \geq 1$ (positive integer), there is a unique field of order $q = p^k$ (up to isomorphism)
Finite fields: $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5, \mathbb{F}_7, \mathbb{F}_8, \mathbb{F}_9, \mathbb{F}_{11}, \mathbb{F}_{13}, \mathbb{F}_{16}, \mathbb{F}_{17}, \ldots$

$\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$

+	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

×	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

$\alpha + \alpha = (1+1)\alpha = 0\alpha = 0$

$\operatorname{char} \mathbb{F}_4 = 2$.

$\mathbb{F}_4 \supset \mathbb{F}_2$ of degree $[\mathbb{F}_4 : \mathbb{F}_2] = 2$ with basis $1, \alpha$

$\mathbb{F}_4 = \{a \cdot 1 + b\alpha : a, b \in \mathbb{F}_2\}$

$= \{0, 1, \alpha, 1+\alpha\}$ where $\alpha^2 = \alpha + 1$.

$= \{0, 1, \alpha, \alpha^2\}$ β

$\mathbb{F}_4 = \mathbb{F}_2[\alpha]$

The minimal poly. of α over $\mathbb{F}_2$ is $x^2 + x + 1$.

Irreducible polynomials over $\mathbb{F}_2 = \{0,1\}$

degree 1: x, $x+1$ (both irreducible)

degree 2: x^2, x^2+1, x^2+x, x^2+x+1
$x^2 = x\cdot x$, $x^2+1 = (x+1)(x+1)$, $x^2+x = x(x+1)$ — reducible; x^2+x+1 irreducible

degree 3: $x^3 = x\cdot x\cdot x$
$x^3+1 = (x+1)(x^2+x+1)$
$x^3+x = x\cdot(x+1)^2$
x^3+x+1 irreducible
$x^3+x^2 = x\cdot x\cdot(x+1)$
x^3+x^2+1 irreducible
$x^3+x^2+x = x(x^2+x+1)$
$x^3+x^2+x+1 = (x+1)^3$

In general the nonzero elements of $\mathbb{F}_q$ form a cyclic group of order $q-1$.

There is only one finite field of each order $q=p^k$ (p prime, $k \geq 1$) up to isomorphism.

If $\mathbb{F}_q$ is a finite field then it must have $\operatorname{char} \mathbb{F}_q = p$ for some prime p.
$|\mathbb{F}_q| = q < \infty$. So $\mathbb{F}_q$ is an extension $\mathbb{F}_q \geq \mathbb{F}_p$ hence a vector space of some dimension k.
Let $\alpha_1, \dots, \alpha_k$ be a basis for $\mathbb{F}_q$ over $\mathbb{F}_p$ i.e. $\mathbb{F}_q = \{a_1\alpha_1 + a_2\alpha_2 + \cdots + a_k\alpha_k : a_1, \dots, a_k \in \mathbb{F}_p\}$.

$$q = |\mathbb{F}_q| = p^k$$

There are 2^n polynomials of degree n and they are all monic: $x^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0$, $c_0, c_1, \dots, c_{n-1} \in \mathbb{F}_2$

Let α be a root of x^2+x+1. The other root is $\alpha+1$.
$\alpha^2+\alpha+1 = 0 \Rightarrow \alpha^2 = -\alpha-1 = \alpha+1$

Note: The roots of $ax^2+bx+c=0$ are $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$ except in characteristic 2.

$\mathbb{F}_8 = \mathbb{F}_2[\gamma]$ where γ is a root of x^3+x+1 i.e. $\gamma^3 = \gamma+1$
$= \{a\cdot 1 + b\cdot\gamma + c\cdot\gamma^2 : a,b,c \in \mathbb{F}_2\}$.
$= \{0, 1, \gamma, \gamma+1, \gamma^2, \gamma^2+1, \gamma^2+\gamma, \gamma^2+\gamma+1\}$.
γ^3 γ^6 γ^4 γ^5

x^3+x+1 has three roots in $\mathbb{F}_8$: $\gamma, \gamma^2, \gamma^4$.
x^3+x^2+1 has three roots in $\mathbb{F}_8$: $\gamma^3, \gamma^5, \gamma^6 = \gamma^7$

$\gamma^0 = 1$
$\gamma^1 = \gamma$
$\gamma^2 = \gamma^2$
$\gamma^3 = \gamma+1$
$\gamma^4 = \gamma^2+\gamma$
$\gamma^5 = \gamma^3+\gamma^2 = \gamma^2+\gamma+1$
$\gamma^6 = \gamma^3+\gamma^2+\gamma = (\gamma+1)+\gamma^2+\gamma = \gamma^2+1$
$\gamma^7 = \gamma^3+\gamma = (\gamma+1)+\gamma = 1$

$\mathbb{F}_9 = \mathbb{F}_3[i]$ compare: $\mathbb{C} = \mathbb{R}[i]$, $\mathbb{Q}[i] \supset \mathbb{Q}$, $i = \sqrt{-1}$. $\{1, i\}$ is a basis of the extension in each case.
$\mathbb{Q}[\sqrt{2}] \supset \mathbb{Q}$

$= \{a+bi : a, b \in \mathbb{F}_3\}$

$= \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$

$\quad\quad = \theta^8 \quad \theta^4 \quad \theta^6 \quad \theta \quad \theta^7 \quad \theta^2 \quad \theta^3 \quad \theta^5$

θ is a primitive element: its powers give all the nonzero elements of $\mathbb{F}_9$.

$i = \sqrt{-1} = \sqrt{2}$ $\quad\quad \mathbb{F}_9 = \mathbb{F}_3[i] = \mathbb{F}_3[\sqrt{2}]$

$\theta^0 = 1$

$\theta^1 = \theta = 1+i$

$\theta^2 = (1+i)^2 = 1 + 2i + i^2 = 2i$

$\theta^3 = \frac{2i}{\theta^2}(1+i) = \frac{2i}{\theta} = -2 + 2i = 1 + 2i$

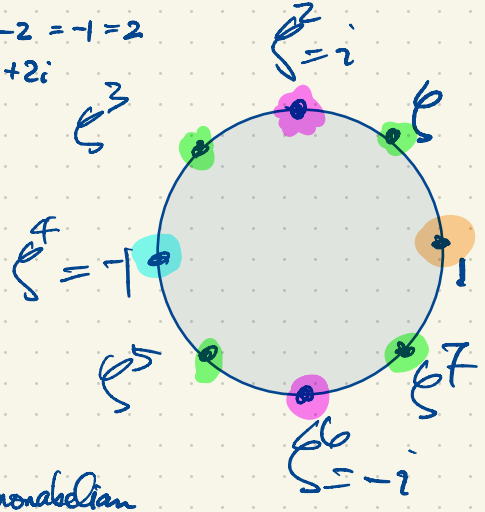
$\theta^4 = \theta^3 \cdot \theta = (1+2i)(1+i) = 1 - 2 = -1 = 2$

$\theta^5 = \theta^4 \cdot \theta = -\theta = 2\theta = 2 + 2i$

$\theta^6 = \theta^4 \cdot \theta^2 = -\theta^2$

$\theta^7 = \theta^4 \cdot \theta^3 = -\theta^3$

$\theta^8 = \theta^4 \cdot \theta^4 = -\theta^4$



Every finite field $\mathbb{F}_q$ ($q = p^k$, p prime) has a primitive element i.e. an element whose powers give all the nonzero field elements.

Why? Idea of proof: Eg. to see that $\mathbb{F}_9$ has a primitive element: The nonzero elements form a multiplicative group of order 8. There are five groups of order 8 up to isomorphism:

- dihedral group of order 8 (symmetry group of a square) } nonabelian
- quaternion " " "
- abelian { C_8 (four elements of order 8, two elements of order 4, one element of order 2)
- abelian { $C_2 \times C_4$ (four elements of order 4, three elements of order 2)
- abelian { $C_2 \times C_2 \times C_2$ (with seven elements of order 2)

Every abelian group is a direct product of cyclic groups.

C_n = cyclic group of order n (multiplicative)

$C_n = \{1, g, g^2, \dots, g^{n-1}\}$, $g^n = 1$.

In a field of order q, the polynomial x^2-1 has at most 2 roots.

(In $F[x]$, where F is any field, every polynomial of degree k has at most k roots.)

If $f(x) \in F[x]$ has k roots $r_1, \dots, r_k \in F$, then $f(x) = (x-r_1)(x-r_2)\cdots(x-r_k)h(x)$

degree k

$x^2-1 = (x-1)(x+1)$

$\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{2}] \neq \mathbb{F}_5[i]$, $i = \sqrt{-1} = \sqrt{4} = \pm 2$

$1, \sqrt{2}$ is a basis

In $\mathbb{F}_5$, -1 is already a square.

$\mathbb{F}_5[i] = \mathbb{F}_5[2] = \mathbb{F}_5$

$\mathbb{Q}[\sqrt{4}] = \mathbb{Q}[2] = \mathbb{Q}$

$\mathbb{R}[\sqrt{2}] = \mathbb{R}$

$\mathbb{R}[i] = \mathbb{C}$

In $\mathbb{R}[x]$, $\begin{cases} x^2-2 \text{ is reducible since } x^2-2 = (x+\sqrt{2})(x-\sqrt{2}). \\ x^2+1 \text{ is irreducible.} \end{cases}$

How do we extend $\mathbb{F}_p$ to $\mathbb{F}_{p^2}$? We want a quadratic extension $[\mathbb{F}_{p^2} : \mathbb{F}_p] = 2$.

A choice of basis is $\{1, \sqrt{a}\}$ if $a \in \mathbb{F}_p$ is not a square of any element in $\mathbb{F}_p$ ie. $x^2-a \in \mathbb{F}_p[x]$ should be irreducible.

When p is an odd prime, there are $p-1$ nonzero elements and half of them are squares, half are nonsquares.

When $p=5$, the nonzero elements of $\mathbb{F}_5$ are $1,2,3,4$ where $1,4$ are squares; $2,3$ are nonsquares.

$\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{2}] = \mathbb{F}_5[\sqrt{3}]$.

When $p=2$, $x^2-a = (x-a)^2$ ie. $x^2 = x \cdot x$ reducible, $x^2-1 = (x-1)^2$ reducible

$\mathbb{F}_2 = \{0,1\}$ has squares only.

But x^2+x+1 is irreducible in $\mathbb{F}_2[x]$

$\mathbb{F}_4 = \mathbb{F}_2[\alpha]$, α a root of x^2+x+1.

If $q = p^k$ then $\mathbb{F}_q \supset \mathbb{F}_p$ is an extension of degree $[\mathbb{F}_q : \mathbb{F}_p] = k$ with exactly k automorphisms.

In $\mathbb{F}_9 = \mathbb{F}_3[i]$, the map $a+bi \mapsto a-bi$ is the nonidentity automorphism.

In $\mathbb{F}_{25} = \mathbb{F}_5[\sqrt{2}]$, the $\cdots$ $a+b\sqrt{2} \mapsto a-b\sqrt{2}$ $\cdots$

$\mathbb{F}_4 = \mathbb{F}_2[\kappa] = \{0, 1, \alpha, \beta\}$ the map $0 \to 0$, $1 \to 1$, $\alpha \to \beta$, $\beta \to \alpha$ $\cdots$

$\alpha + 1 = \alpha^2$

Finite fields are Galois extensions of their prime fields: $\mathbb{F}_q \supseteq \mathbb{F}_p$, $q = p^k$, p prime

$[\mathbb{F}_q : \mathbb{F}_p] = k$ so $G = \operatorname{Aut} \mathbb{F}_q$ has order $|G| = k$ and $G = \{1, \sigma, \sigma^2, \cdots, \sigma^{k-1}\}$, $\sigma^k = 1$. Here $\sigma(x) = x^p$.

$\sigma(xy) = (xy)^p = x^p y^p = \sigma(x)\sigma(y)$ for all $x, y \in \mathbb{F}_q$.

$\sigma(x+y) = (x+y)^p = x^p + px^{p-1}y + \frac{p(p-1)}{2}x^{p-2}y^2 + \cdots + pxy^{p-1} + y^p$ by the Binomial Theorem $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i$

where $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, $n! = 1 \times 2 \times 3 \times \cdots \times n$

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$

$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$

$\binom{n}{0} = \frac{n!}{0!\,n!} = 1 = \binom{n}{n}$

divisible by p

$= x^p + y^p = \sigma(x) + \sigma(y)$

$\sigma : \mathbb{F}_q \to \mathbb{F}_q$ is a homomorphism. All elements of $\mathbb{F}_q$ are roots of $x^q - x$.

$\ker \sigma = \{x \in \mathbb{F}_q : \sigma(x) = 0\} = \{0\}$ ($x^p = 0$) so σ is one-to-one.

Since $\mathbb{F}_q$ is finite, σ is onto. So σ is an isomorphism $\mathbb{F}_q \to \mathbb{F}_q$ i.e. σ is an automorphism of $\mathbb{F}_q$.

$\operatorname{Aut} \mathbb{F}_q \supseteq \{1, \sigma, \sigma^2, \sigma^3, \cdots\}$ but these automorphisms can't all be distinct

$\sigma^k(x) = \underbrace{\sigma(\sigma(\sigma(\cdots(\sigma(x))\cdots)))}_{k \text{ times}} = \underbrace{(((x^p)^p)^p)\cdots)^p}_{k \text{ times}} = x^{p^k} = x^q = x$

$\sigma^k = 1$

In $\mathbb{F}_q^\times = \{x \in \mathbb{F}_q : x \neq 0\}$ is a multiplicative group (actually cyclic) of order $q-1$. $x^{q-1} = 1$ for all $x \in \mathbb{F}_q^\times$

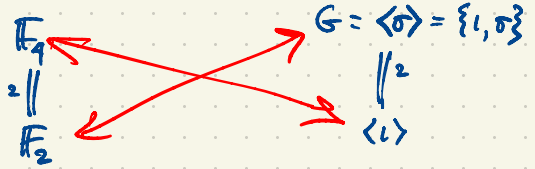
Eg. $\mathbb{F}_4 > \mathbb{F}_2$ of degree $[\mathbb{F}_4 : \mathbb{F}_2] = 2$ with basis $\{1, \alpha\}$

$\mathbb{F}_4 = \{0, 1, \alpha, \beta\}$ where $\beta = \alpha^2 = \alpha + 1$

$= \{a \cdot 1 + b \cdot \alpha : a, b \in \mathbb{F}_2\}$

$\mathrm{Aut}\, \mathbb{F}_4 = \langle \sigma \rangle = \{\iota, \sigma\}$.

x	$\sigma(x) = x^2$	$\sigma^2(x) = x^4$
0	0	0
1	1	1
α	β	α
β	α	β



Eg. $\mathbb{F}_9 \supset \mathbb{F}_3 = \{0, 1, 2\}$, $[\mathbb{F}_9 : \mathbb{F}_3] = 2$ with basis $\{1, i\}$

$\mathbb{F}_9 = \{a + bi : a, b \in \mathbb{F}_3\}$

$i = \sqrt{-1} = \sqrt{2}$

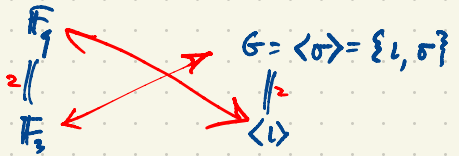
$\sigma(x) = x^3$

$\sigma(a + bi) = a - bi$ for $a, b \in \mathbb{F}_3$

x	$\sigma(x) = x^3$
0	0
1	1
2	2
i	$-i = 2i$
$2i$	$-2i = i$
$a + bi$	$a - bi$

$(a + bi)^3 = a^3 + 3a^2bi + 3a(bi)^2 + (bi)^3$

$= a - bi$



Eg. $\mathbb{F}_8 \supset \mathbb{F}_2 = \{0, 1\}$, $[\mathbb{F}_8 : \mathbb{F}_2] = 3 = |G|$ where $G = \mathrm{Aut}\, \mathbb{F}_8 = \langle \sigma \rangle = \{\iota, \sigma, \sigma^2\}$, $\sigma^3 = 1$

$\mathbb{F}_8 = \{a + b\gamma + c\gamma^2 : a, b, c \in \mathbb{F}_2\}$, $\gamma^3 = \gamma + 1$

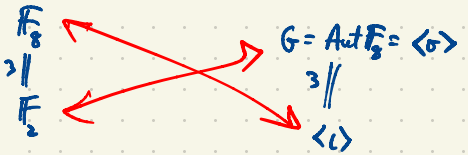
$\{1, \gamma, \gamma^2\}$ basis

$\sigma(x) = x^2$,

$\sigma^2(x) = (x^2)^2 = x^4$

$\sigma^3(x) = ((x^2)^2)^2 = x^8 = x$

x	$\sigma(x) = x^2$
0	0
1	1
γ	γ^2
γ^2	$\gamma^4 = \gamma + \gamma^2$
$\gamma^3 = 1 + \gamma$	$\gamma^6 = 1 + \gamma^2$
$\gamma^4 = \gamma + \gamma^2$	γ
$\gamma^5 = \gamma^2 + \gamma + 1$	$\gamma^3 = 1 + \gamma$
$\gamma^6 = 1 + \gamma^2$	$\gamma^5 = \gamma^2 + \gamma + 1$
$\gamma^7 = 1$	1



If $f(x) \in F[x]$ is irreducible, then we say any two roots α, β of $f(x)$ (typically in an extension field $E \supseteq F$) then α, β are conjugates.

Eg. $f(x) = x^2 - 2 \in \mathbb{Q}[x]$ has roots $\pm\sqrt{2} \in \mathbb{R}$ or in $\mathbb{Q}[\sqrt{2}]$. $\pm\sqrt{2}$ are conjugates.

If $f(x) = x^2 + 1 \in \mathbb{Q}[x]$ has roots $\pm i \in \mathbb{C}$ or $\mathbb{Q}[i]$. $\pm i$ are conjugates.

In E there can be an automorphism $\sigma \in \mathrm{Aut}\, E$ fixing every element of F and mapping a root of $f(x)$ to any of its conjugates.

Eg. $f(x) = x^3 - 2$ has three roots $\alpha, \alpha\omega, \alpha\omega^2$ where $\alpha = \sqrt[3]{2}$, $\omega = e^{2\pi i/3} = \frac{-1+\sqrt{-3}}{2}$, $\omega^2 = e^{4\pi i/3} = \frac{-1-\sqrt{-3}}{2}$.

The elements $\alpha, \alpha\omega, \alpha\omega^2$ are conjugates. These are all the conjugates of α.

in $\mathbb{Q}[\alpha, \omega] \supset \mathbb{Q}$, $[\mathbb{Q}[\alpha,\omega] : \mathbb{Q}] = 6$.

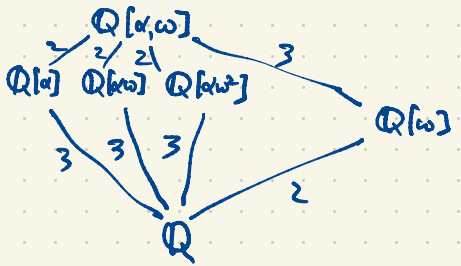
$x^3 - 2 = (x-\alpha)(x-\alpha\omega)(x-\alpha\omega^2)$

$\mathbb{Q}[\alpha,\omega]$ is the splitting field of $f(x) = x^3 - 2$

$\mathbb{Q}[\alpha]$ is not the splitting field of $f(x) = x^3 - 2 = (x-\alpha)(x^2 + \alpha x + \alpha^2)$

$[\mathbb{Q}[\alpha] : \mathbb{Q}] = 3$

$[\mathbb{Q}[\alpha\omega] : \mathbb{Q}] = 3$



Eg. $\mathbb{F}_8 \supset \mathbb{F}_2 = \{0,1\}$, $[\mathbb{F}_8 : \mathbb{F}_2] = 3 = |G|$ where $G = \mathrm{Aut}\,\mathbb{F}_8 = \langle \sigma \rangle = \{1, \sigma, \sigma^2\}$, $\sigma^3 = 1$

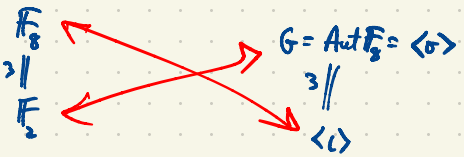
$\mathbb{F}_8 = \{a + b\gamma + c\gamma^2 : a,b,c \in \mathbb{F}_2\}$, $\gamma^3 = \gamma + 1$

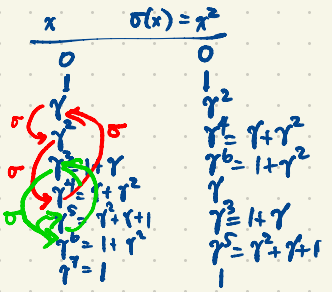
$\{1, \gamma, \gamma^2\}$ basis

$\sigma(x) = x^2$,

$\sigma^2(x) = (x^2)^2 = x^4$

$\sigma^3(x) = ((x^2)^2)^2 = x^8 = x$





$f(x) = x^3 + x + 1 \in \mathbb{F}_2[x]$ is irreducible

It has roots in $\mathbb{F}_8$: $\gamma, \gamma^2, \gamma^4$

$\sigma(\gamma^4) = \gamma^8 = \gamma$ $\quad \sigma(\gamma^5) = \gamma^{10} = \gamma^3$

$\sigma(\gamma^3) = \gamma^6$

$\sigma(\gamma^6) = \gamma^{12} = \gamma^5$

$f(x) = x^3 + x + 1 = (x - \gamma)(x - \gamma^2)(x - \gamma^4)$

$(\gamma^3 + \gamma + 1)^2 = 0$

$\gamma^6 + \gamma^2 + 1 = 0$

$\gamma^3 \in \mathbb{F}_8$ must have minimal poly. $g(x) \in \mathbb{F}_2[x]$ of degree 3. This must be $g(x) = x^3 + x^2 + 1$

so $g(x) = x^3 + x^2 + 1$ must have roots $\gamma^3, \gamma^6, \gamma^5$

The roots of $x^8 - x \in \mathbb{F}_2[x]$ are all the eight elements of $\mathbb{F}_8$.

$$x^8 - x = x(x^7 - 1) = x(x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$
$$= x(x+1)(x^3 + x + 1)(x^3 + x^2 + 1)$$
$$0 \quad 1 \quad \gamma, \gamma^2, \gamma^4 \quad \gamma^3, \gamma^5, \gamma^6$$

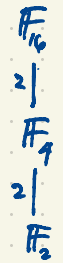
$\mathbb{F}_5$: all elements are roots of $x^5 - x = x(x^4-1) = x(x^2+1)(x^2-1) = x(x-2)(x-3)(x-1)(x+1)$

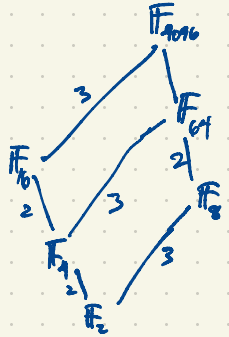
$= x(x-1)(x-2)(x-3)(x-4)$

0 1 2 3 4

Subfields of $\mathbb{F}_{16}$: $\mathbb{F}_2$, $\mathbb{F}_4$, $\mathbb{F}_{16}$

$[\mathbb{F}_8 : \mathbb{F}_2] = 3$





Math 4550 Spring 2025 = 45^2 Theory of Numbers

Putnam Exam 2024 Dec 7 8:30 am – 1:30 pm

Interested? Email me with 'Putnam' in subject line.

More examples of fields: $F((x)) \supset F(x) \supset F$ where F is a field.

$F((x))$: Laurent series in x with coefficients in F

$F(x)$: rational functions in x with coefficients in F

x is an indeterminate (a symbol)

Eg. $f(x) = \frac{x}{1-x-x^2} \in \mathbb{Q}(x)$ can be regarded as an infinite series in x with coefficients in $\mathbb{Q}$

$= F_0 + F_1 x + F_2 x^2 + F_3 x^3 + \cdots$ where $F_i \in \mathbb{Q}$

$$f'(x) = \frac{(1-x-x^2)1 - x(-1-2x)}{(1-x-x^2)^2} = \frac{1+x^2}{(1-x-x^2)^2}$$

$$f''(x) = \frac{(1-x-x^2)^2(2x) - (1+x^2)2(1-x-x^2)(-1-2x)}{(1-x-x^2)^4} = \frac{(1-x-x^2)(2x) + 2(1+x^2)(1+2x)}{(1-x-x^2)^3} = \frac{2x-2x^2-2x^3+2(1+2x+x^2+2x^3)}{(1-x-x^2)^3}$$

$$= \frac{2+6x+2x^3}{(1-x-x^2)^3}$$

$f'''(x) =$ etc.

$f^{(4)}(x) = f^{IV}(x) =$ etc.

Taylor series centered at 0 for $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \frac{f^{IV}(0)}{24}x^4 + \cdots$

$= 0 + 1x + \frac{2}{2}x^2 + \frac{12}{6}x^3 + \frac{72}{24}x^4 + \cdots$

$= x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + 13x^7 + \cdots$

The Fibonacci sequence F_n is defined recursively by

$$F_n = \begin{cases} 0, & \text{if } n=0 \\ 1, & \text{if } n=1 \\ F_{n-1} + F_{n-2}, & \text{if } n \geq 2 \end{cases}$$

Alternatively: $f(x) = \frac{x}{1-x-x^2} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$

($a_0 = 0$, $a_1 = 1$, $a_2 = 1$)

$$x = (1-x-x^2)(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots)$$

$$= a_0 + (a_1 - a_0)x + (a_2 - a_1 - a_0)x^2 + (a_3 - a_2 - a_1)x^3 + (a_4 - a_3 - a_2)x^4 + \cdots$$

$a_0 = 0$, $a_1 - a_0 = 1$, $a_2 - a_1 - a_0 = 0$, $a_3 - a_2 - a_1 = 0$, $a_4 - a_3 - a_2 = 0$

Third way: $\frac{1}{1-u} = 1 + u + u^2 + u^3 + u^4 + \cdots$ (geometric series)

Since $(1-u)(1+u+u^2+u^3+u^4+\cdots) = 1 - u + u - u^2 + u^2 - u^3 + u^3 + \cdots = 1$

Substitute $u = x + x^2$

$$\frac{x}{1-x-x^2} = x\left(1 + (x+x^2) + (x+x^2)^2 + (x+x^2)^3 + (x+x^2)^4 + \cdots\right)$$

$$= x\left(1 + (x+x^2) + (x^2+2x^3+x^4) + (x^3+3x^4+3x^5+x^6) + (x^4+4x^5+6x^6+4x^7+x^8) + \cdots\right)$$

$$= x\left(1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots\right)$$

$$= x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$$

Fourth method:

$$\frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{A}{1-\alpha x} + \frac{B}{1-\beta x} \Rightarrow x = A(1-\beta x) + B(1-\alpha x) \Rightarrow$$

(for $x = \frac{1}{\alpha}$) $\frac{1}{\alpha} = A\left(1 - \frac{\beta}{\alpha}\right) \Rightarrow 1 = A(\alpha - \beta) \Rightarrow A = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}$

(for $x = \frac{1}{\beta}$) $\frac{1}{\beta} = B\left(1 - \frac{\alpha}{\beta}\right) \Rightarrow 1 = B(\beta - \alpha) = B(-\sqrt{5}) \Rightarrow B = -\frac{1}{\sqrt{5}}$

α, β are the reciprocal roots of $1 - x - x^2 = x^2(x^{-2} - x^{-1} - 1)$

$\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618$, $\beta = \frac{1-\sqrt{5}}{2} \approx -0.618$, $\alpha - \beta = \sqrt{5}$

$$\frac{x}{1-x-x^2} = \frac{x}{(1-\alpha x)(1-\beta x)} = \frac{1}{\sqrt{5}}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right) = \frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty} \alpha^n x^n - \sum_{n=0}^{\infty} \beta^n x^n\right) = \frac{1}{\sqrt{5}} \sum_{n=0}^{\infty} (\alpha^n - \beta^n) x^n = \sum_{n=0}^{\infty} F_n x^n = x + x^2 + 2x^3 + 3x^4 + 5x^5 + \cdots$$

where $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$

$\frac{F_{n+1}}{F_n} \to \alpha$ $\quad F_n \sim \frac{1}{\sqrt{5}} \cdot \alpha^n$ $\quad F_n = \frac{\alpha^n}{\sqrt{5}}$ rounded to the nearest integer.

$|\beta| < 1$ so $\beta^n \to 0$ decays exponentially

$|\alpha| > 1$ so $\alpha^n \to \infty$ grows

Eg. Count the number a_n of sequences of 0's and 1's of length n having no two consecutive 1's.

n		
0	""	$a_0 = 1$
1	'0', '1'	$a_1 = 2$
2	00, 10, 01	$a_2 = 3$
3	000, 100, 010, 001, 101	$a_3 = 5$
4	----------	$a_4 = 8$

Other series are relevant in combinatorial applications in which $f(x)$ cannot converge anywhere eg.

$f(x) = \sum_{n=0}^{\infty} n!\, x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \cdots$

$f(x)^2 = (1 + x + 2x^2 + 6x^3 + \cdots)^2 = 1 + 2x + 5x^2 + \cdots$

$\frac{f(x)}{x} = \frac{1}{x} + 1 + 2x + 6x^2 + 24x^3 + \cdots$

$\frac{x}{1 - x - x^2} = x + x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots$

$\frac{1}{1 - x - x^2} = 1 + x + 2x^2 + 3x^3 + \cdots$

$\frac{1}{x - x^2 - x^3} = \frac{1}{x} + 1 + 2x + 3x^2 + 5x^3 + \cdots$

What is a series $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$ really?
We think of f as the sequence $(a_0, a_1, a_2, a_3, \cdots)$
So if $g(x) = b_0 + b_1 x + b_2 x^2 + \cdots$ then g is really g is $(b_0, b_1, b_2, b_3, \cdots)$

$f + g = (a_0 + b_0,\ a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3,\ \cdots)$ entrywise addition

$fg = (a_0 b_0,\ a_0 b_1 + a_1 b_0,\ a_0 b_2 + a_1 b_1 + a_2 b_0,\ a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0, \cdots)$ $\quad fg = (c_0, c_1, c_2, c_3, \cdots)$, $c_n = \sum_{k=0}^{n} a_k b_{n-k}$

multiplication is by convolution (not entrywise)

$F[[x]]$ = power series in x with coefficients in F (a ring)

$F((x))$ = field of quotients of $F[[x]]$ (like $F[[x]]$ but with some negative powers of x)

eg. $F = \mathbb{F}_2 = \{0,1\}$

$$\frac{x^2+x^4+x^5+x^7+\cdots}{x^5+x^6+x^9+x^{11}+\cdots} = x^{-3}+x^{-2}+1+\cdots \quad \text{(higher degree terms)}$$

$$x^2+x^4+x^5+x^7+\cdots = (x^5+x^6+x^9+x^{11}+\cdots)(x^{-3}+x^{-2}+1+\underbrace{0}_{x^2}x^2\cdots)$$

$1x^{-2}$ $\quad 0x^{-1}$ $\quad x^0=1$ $\quad 0x$ $\quad 0x^2$

In the ring $F[[x]]$, the units (ie. invertible elements) are all the elements with nonzero constant term.

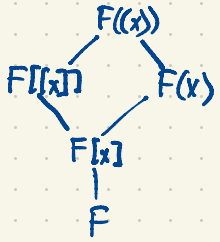
$F((x))$ is however a field i.e. all nonzero elements are units

Why can't we allow infinitely many powers of x with negative exponents as well as positive exponents?

$(\cdots+x^{-3}+x^{-2}+x^{-1}+1+x+x^2+x^3+\cdots)(\cdots+5x^{-3}+2x^{-2}+7x^{-1}+11+13x+2x^2+3x^3+x^4+\cdots)$ is undefined whereas

$$(x^{-2}+x^{-1}+1+x+x^2+x^3+\cdots)(7x^{-1}+11+13x+2x^2+3x^3+x^4+\cdots) = 7x^{-3}+18x^{-2}+31x^{-1}+33+36x+\cdots$$

$$\frac{1}{x^2+x^3} \overset{\in F(x)}{=} \frac{1}{x^2(1+x)} = \frac{1}{x^2}(1-x+x^2-x^3+x^4-x^5+\cdots) = x^{-2}-x^{-1}+1-x+x^2-x^3+x^4-x^5+\cdots$$



Automorphisms of $\mathbb{Q}(x) \supset \mathbb{Q}$ includes $f(x) \longmapsto f(x+1)$ has inverse $f(x) \longmapsto f(x-1)$.

$f(x)+g(x) \longmapsto f(x+1) + g(x+1)$

$f(x)g(x) \longmapsto f(x+1)\, g(x+1)$

$[\mathbb{Q}(x) : \mathbb{Q}] = \infty$ (actually $\aleph_0$)

This is a start on one of the HW4 problems.

How about square roots?

For $f(x) \in \mathbb{Q}(x)$, when does $\sqrt{f(x)} \in \mathbb{Q}(x)$?

$\sqrt{x} \notin \mathbb{Q}(x)$. Very small fraction of functions $f(x) \in \mathbb{Q}(x)$ have $\sqrt{f(x)} \in \mathbb{Q}(x)$.

eg. if $F = \mathbb{Q}(x)$ then $E = F(\sqrt{x}) = \mathbb{Q}(x, \sqrt{x}) = \mathbb{Q}(\sqrt{x})$ since $x \in \mathbb{Q}(\sqrt{x})$, in fact $x \in \mathbb{Q}[\sqrt{x}]$.

$\sqrt{1+x} \in \mathbb{Q}((x))$

$\sqrt{1+x} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$, $a_i \in \mathbb{Q}$

$\in \mathbb{Q}[[x]]$

$1+x = (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)^2 = a_0^2 + 2a_0a_1 x + (a_1^2 + 2a_0a_2)x^2 + (2a_0a_3 + 2a_1a_2)x^3 + (a_2^2 + 2a_1a_3 + 2a_0a_4)x^4 + \cdots$

$a_0 = \pm 1$. Let's take $a_0 = 1$. $= 1 + 2a_1 x + (a_1^2 + 2a_2)x + (2a_3 + 2a_1a_2)x^3 + (a_2^2 + 2a_1a_3 + 2a_4)x^4 + \cdots$

Now $a_1 = \frac{1}{2}$ $= 1 + x + \underbrace{(\tfrac{1}{4} + 2a_2)}_{0} x^2 + \underbrace{(2a_3 + a_2)}_{0} x^3 + (a_2^2 + a_3 + 2a_4)x^4 + \cdots$

$a_2 = -\frac{1}{8}$

$a_3 = \frac{1}{16}$

etc.

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

The two square roots of $1+x$ in $\mathbb{Q}[[x]]$ are $\pm\sqrt{1+x} = \pm(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots)$

Binomial Theorem $(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k$ $\quad \binom{a}{k} = \frac{a(a-1)(a-2)\cdots(a-k+1)}{k(k-1)(k-2)\cdots 2\cdot 1} \leftarrow k!$ $\quad k \in \{0,1,2,\dots\}$ $\quad a \in \mathbb{R}$.

(Polynomial of degree k in a; but if $a \in \{0,1,2,\dots\}$ then this value $\binom{a}{k}$ is entry k in row a of Pascal's triangle).

$\sqrt{1+x} =$

$(1+x)^{\frac{1}{2}} = 1 + \underbrace{\frac{\frac{1}{2}}{1}}_{\binom{\frac{1}{2}}{1}} x + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)}{2\cdot 1}}_{\binom{\frac{1}{2}}{2}} x^2 + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3\cdot 2\cdot 1}}_{\binom{\frac{1}{2}}{3}} x^3 + \underbrace{\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4\cdot 3\cdot 2\cdot 1}}_{\binom{\frac{1}{2}}{4}} x^4 + \cdots$

$-\frac{5}{8}\cdot\frac{1}{16} = -\frac{5}{128}$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots \in \mathbb{Q}[[x]]$

$\sqrt{2+x} = \sqrt{2}\sqrt{1+\frac{x}{2}} \in \mathbb{R}[[x]]$ $\quad \notin \mathbb{Q}[[x]]$

$\mathbb{R}$ $\quad \mathbb{Q}[[x]]$

On HW4 about #3 (?)

$\sqrt{4+x} = 2\sqrt{1+\frac{x}{4}} \in \mathbb{Q}[[x]]$

If $q = p^k$, p prime $k \geq 1$ we have a field of order q, unique up to isomorphism, denoted $F = \mathbb{F}_q$.

First suppose q is odd i.e. $p = \operatorname{char} F$ is odd.

Half the nonzero elements are squares.

"ABLE WAS I ERE I SAW ELBA"

Eg. $\mathbb{F}_7 = \{0,1,2,3,4,5,6\}$.

a	a^2
0	0
1	1
2	4
3	2
$-3=4$	2
$-2=5$	4
$-1=6$	1

a	a^2
0	0
± 1	1
± 2	4
± 3	2

$\mathbb{F}_{49} = \mathbb{F}_7[\sqrt{3}] = \mathbb{F}_7[\sqrt{5}] = \mathbb{F}_7[\sqrt{6}] = \mathbb{F}_7[i]$, $i = \sqrt{-1}$

$x^2 - a$ is reducible iff a is a square.

$x^2 - a$ is irreducible iff $F[\sqrt{a}] \supset F$ is a quadratic extension.

If $q = 2^k$ then every element of $\mathbb{F}_q$ has a unique square root and $x \mapsto \sqrt{x}$ is an automorphism of $\mathbb{F}_2$.

$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{R}^*$ $\quad \mathbb{R}$: reals, $\mathbb{R}^*$: hyperreals

Back to basics:

Let R be an integral domain i.e. commutative ring with identity 1 having no zero divisors (i.e. $ab \neq 0 \Rightarrow a \neq 0$ and $b \neq 0$)

eg. $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{R}[x]$ An ideal in R is a subset (actually subring) which is closed under taking R-linear combinations. A subset $A \subseteq R$ with $0 \in A$ is an ideal if

$$r_1 a_1 + r_2 a_2 + \cdots + r_k a_k \in A \quad \text{for all } a_1, \cdots, a_k \in A ; \quad r_1, \cdots, r_k \in R .$$

Eg. if we fix $a_1, \cdots, a_n \in R$ then the ideal generated by $a_1, \cdots, a_n$ is

$$(a_1, \cdots, a_n) = \{ r_1 a_1 + \cdots + r_n a_n : r_1, \cdots, r_n \in R \}$$

(Compare: the span of a set of vectors in a vector space is a subspace).

Eg. in $\mathbb{Z}$, fix an integer m. The principal ideal generated by m is

$(m) = \{ rm : r \in \mathbb{Z} \} = \{ \cdots, -2m, -m, 0, m, 2m, 3m, \cdots \}$ is an ideal in $\mathbb{Z}$.

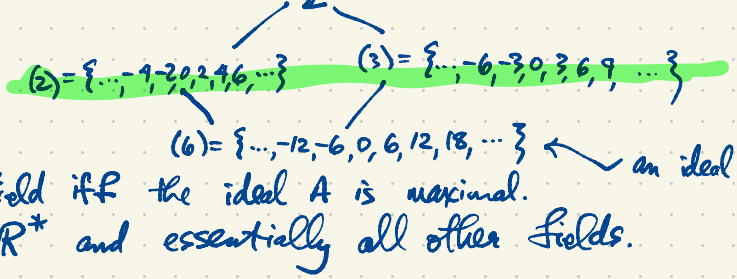
The quotient ring is $\mathbb{Z}/(m) = \mathbb{Z}/m\mathbb{Z} = \{$cosets of (m) in $\mathbb{Z}\} = \{ (m), 1+(m), 2+(m), \cdots, m-1+(m) \}$

Informally $\mathbb{Z}/m\mathbb{Z} = \{ \bar{0}, \bar{1}, \bar{2}, \cdots, \overline{m-1} \}$ or more informally $\{0, 1, 2, \cdots, m-1\}$

$\mathbb{Z}/(p)$ is a field. $\mathbb{Z}/(m)$ is not a prime unless m is prime.

$\mathbb{Z}/(6) = \{ \bar{0}, \bar{1}, \cdots, \bar{5} \}$ is not a field. $\bar{2} \cdot \bar{3} = \bar{6} = \bar{0}$ (zero divisors $\bar{2}, \bar{3}, \bar{4}$; units $\bar{1}, \bar{5} = -\bar{1}$).

$\mathbb{Z}/(6)$ fails to be a field because the ideal (6) is not maximal; it is contained in (2) and (3).



examples of maximal ideals (not contained in any larger ideals)

an ideal which is not maximal (it's contained in larger ideals).

$\mathbb{Z}/(2) = \mathbb{F}_2$ and $\mathbb{Z}/(3) = \mathbb{F}_3$ are fields.

The quotient ring R/A is a field iff the ideal A is maximal.

We use this to construct $\mathbb{R}$, $\mathbb{R}^*$ and essentially all other fields.

eg. $\mathbb{Z}[x]$ has many examples of subrings and ideals.

eg. $(x^2+1) \subset \mathbb{Z}[x]$ $\quad (x^2+1) = \{h(x)(x^2+1) : h(x) \in \mathbb{Z}[x]\}$

$\mathbb{Z}[x]/(x^2+1) = \{a+bx + (x^2+1) : a,b \in \mathbb{Z}\} \cong \{a+bi : a,b \in \mathbb{Z}\} = \mathbb{Z}[i]$.

"Gaussian integers"

The ideal (x^2+1) is not maximal.

Not a field. The only units (invertible elements) are $1, -1, i, -i$

We have a homomorphism $\phi: \mathbb{Z}[x] \longrightarrow \mathbb{Z}[i]$ $\quad f(x) \longmapsto f(i)$ (evaluate at i)

which is onto. Its kernel is $\ker \phi = (x^2+1)$.

The first isomorphism theorem for rings: $\mathbb{Z}[x]/(x^2+1) \cong \mathbb{Z}[i]$

($\mathbb{Z}[x]$: domain of ϕ; (x^2+1): kernel of ϕ; $\mathbb{Z}[i]$: image of ϕ)

$(3) = \{3h(x) : h(x) \in \mathbb{Z}[x]\}$ is also an ideal of $\mathbb{Z}[x]$.

$\mathbb{Z}[x]/(3) \cong \mathbb{F}_3[x]$, $\quad \mathbb{F}_3 = \{0,1,2\}$ $\quad \mathbb{F}_3[x]$ is a ring but not a field. (3) is not a maximal ideal.

$\mathbb{Z}[x]/(3, x^2+1) \cong \mathbb{F}_9 = \{a+bi : a,b \in \mathbb{F}_3\}$.

is a field

$\mathbb{Z}[x]$

$(3, x^2+1) = \{3h(x) + (x^2+1)g(x) : h(x), g(x) \in \mathbb{Z}[x]\}$.

(3) $\quad (x^2+1)$ $\quad$ non-maximal ideals

Construction of $\mathbb{R}$ from $\mathbb{Q}$ (one way)

$\mathbb{Q}^\infty = \{(a_0, a_1, a_2, \cdots) : a_i \in \mathbb{Q}\}$ is a ring with coordinatewise addition, multiplication, subtraction

$(1,1,1,1,\cdots)(a_0,a_1,a_2,\cdots) = (a_0,a_1,a_2,a_3,\cdots)$ commutative ring with identity

$(1,0,1,0,1,0,\cdots)(0,1,0,1,0,1,\cdots) = (0,0,0,0,0,0,\cdots) = 0$ zero divisors. $\mathbb{Q}^\infty$ is not a field.

$Z = \{(z_0, z_1, z_2, \cdots) \in \mathbb{Q}^\infty :$ for all n there exists M such that $\underbrace{-\tfrac{1}{n} < z_k < \tfrac{1}{n}}_{|z_k| < \frac{1}{n}}$ whenever $k > M\}$.

i.e. $z_k \to 0$ as $k \to \infty$. (The sequences of rationals having $\lim_{k\to\infty} z_k = 0$)

This is a subring $Z \subset \mathbb{Q}^\infty$.

A larger subring $R \subset \mathbb{Q}^\infty$ is the subring of Cauchy sequences i.e.

$R = \{r = (r_0, r_1, r_2, \cdots) \in \mathbb{Q}^\infty$ such that for all n there exists M such that $|r_k - r_\ell| < \frac{1}{n}$ whenever $k, \ell > M\}$.

$R \subset \mathbb{Q}^\infty$ is also a subring. (commutative ring with identity)

$Z \subset R$ is a subring, in fact Z is an ideal in R

$R/Z \cong \mathbb{R}$

There is a field extension $\mathbb{R}^* \supset \mathbb{R}$ having infinitesimals and infinite elements.

In $\mathbb{R}^*$ there exist elements $\varepsilon \in \mathbb{R}^*$ such that $\varepsilon > 0$ but $\varepsilon < \frac{1}{n}$ for $n = 1, 2, 3, \cdots$