University of Wyoming

Math 4520—Fall 2024

Algebra III

Fields

Department of Mathematics

Sample Exam

December, 2024

This sample exam is intended to resemble the Exam (8:00–10:00 am on Wednesday, December 11, 2024 in our usual lecture room, CR 141) in approximate length, difficulty, and style, although clearly the content may differ. The actual content will consist of all material covered in class this semester, and all related handouts. Somewhat greater weight will be placed on the later material (covered after the Test).

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 120 minutes. Total value of questions: 100 points (plus 14 bonus points).

1. *(12 points)* Consider the extension $E \supset F$ where $E = \mathbb{F}_{25}$ and $F = \mathbb{F}_5$, and let $\alpha \in E$ be a root of $x^2 + 2x + 3 \in F[x]$. Find the minimal polynomial of $\beta = 2\alpha + 1$ over F. Simplify your answer.

2. *(12 points)* Determine the minimal polynomial of $\alpha = 2^{1/3} + 2^{2/3}$ over $\mathbb{Q}$.

3. *(12 points)* Let $F = \mathbb{F}_7$, and let $F^{3\times 3}$ be the ring of all 3×3 matrices with entries in F. Find an explicit subring $R \subset F^{3\times 3}$ such that $R \cong \mathbb{F}_{343}$ where $343 = 7^3$. Justify your answer.

   (*Hint:* Show that the polynomial $x^3 - 2 \in F[x]$ is irreducible, and let α be a root of this polynomial in a cubic extension. Then $\mathbb{F}_{343} = F[\alpha]$. We can represent α by a 3×3 matrix over F having the right characteristic polynomial.)

4. *(12 points)* Give an explicit construction of a field of order 16. Justify your answer.

5. *(12 points)* Let $F = \mathbb{F}_p$ where p is prime, and consider $m(x) = x^p - x + 1$. You may use the fact that $m(x)$ is irreducible in $F[x]$.

   (a) The extension $E = F[\alpha]$ has degree $[E : F] = p$ and order $|E| = p^p$, so that $E = \mathbb{F}_q$ where $q = p^p$. Show that $m(x)$ has p distinct roots in E. Express these roots of $m(x)$ in E, expressed as simply as possible.

   (b) The automorphism group $G = \operatorname{Aut} E$ is cyclic of order p generated by $\sigma : E \to E$ where $\sigma(a) = a^p$. How does σ permute the p roots of $m(x)$? (This action has a very simple description.)

6. *(12 points)* Are the fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{3}]$ isomorphic? Justify your answer.

7. *(12 points)* Consider the extension $E \supset F$ where $E = \mathbb{Q}((t))$ and $F = \mathbb{Q}(t)$.

   (a) The rational function $f(t) = \frac{t}{4-t^2} \in F$ can be expanded as a power series in E. Explicitly determine the first five nonzero terms of this series expansion.

   (b) Give an explicit example of an element in E which is not in F, thereby showing that the extension has degree $[E : F] > 1$, i.e. the extension $E \supset F$ is a proper extension.

8. *(30 points)* Answer TRUE or FALSE to each of the following statements.

   (a) If F is any field, then the field of rational functions $F(t)$ has the same characteristic as F. ______*(True/False)*

   (b) If $K \supseteq \mathbb{Q}$ is a finite extension field, then every automorphism of K is continuous. ______*(True/False)*

   (c) For every finite field F of order q, the nonzero elements of F form a multiplicative group of order $q-1$ which is cyclic. ______*(True/False)*

   (d) There is an automorphism of $\mathbb{R}$ mapping $\sqrt{2} \mapsto -\sqrt{2}$. ______*(True/False)*

   (e) There exist subfields $K, L \subseteq \mathbb{R}$ such that $K \neq L$ but $K \cong L$. ______*(True/False)*

   (f) There exists a proper extension field $E \supset \mathbb{C}$. (Recall that 'proper' means $E \neq \mathbb{C}$.) ______*(True/False)*

   (g) Let $E \supseteq \mathbb{Q}$ be an extension field. Then every automorphism of E is a linear transformation of the vector space E over the ground field $\mathbb{Q}$.______*(True/False)*

   (h) If $a \in \mathbb{C}$ is transcendental over $\mathbb{Q}$, then necessarily so are $a+1$ and a^2. ______*(True/False)*

   (i) A regular 60-gon can be constructed using only a straightedge and compass. ______*(True/False)*

   (j) Let $E \supset \mathbb{Q}$ be a cubic field extension, so that $[E : \mathbb{Q}] = 3$. Suppose that E has three distinct automorphisms ι, σ, σ^2. Then $a + \sigma(a) + \sigma^2(a) \in \mathbb{Q}$ for every $a \in E$. ______*(True/False)*