University of Wyoming — Math 4520—Fall 2024 — Algebra III: Fields — Department of Mathematics

Solutions to HW3

1. (a) No, this maps $0 \mapsto 3$. (Automorphisms of fields must map $0 \mapsto 0$.)
   (b) No, this is a homomorphism. However, it is not one-to-one; for example, it maps $x \to x^2$ and also $-x \mapsto x^2$. It is also not onto; for example, x is not in the image of this homomorphism.)
   (c) No, again, this is a homomorphism but it is not onto. The element x is not in its image. (Every polynomial in the image has degree divisible by 3.)
   (d) Yes, this is an automorphism. In fact, its inverse is itself.
   (e) Yes, this is an automorphism; its inverse is $f(x) \mapsto f\big(\frac{3x-5}{2-x}\big)$.

2. Writing $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$, we have

$$x^2 + 3x + 4 = f(x)^2 = a_0^2 + 2a_0a_1x + (a_1^2{+}2a_0a_2)x^2 + (2a_1a_2{+}2a_0a_3)x^3 + \cdots,$$
$$\text{so}\ \ f(x) = 2 + \tfrac{3}{4}x + \tfrac{7}{64}x^2 - \tfrac{21}{512}x^3 + \cdots.$$

   The coefficients were found recursively. The negative of this entire series is also acceptable. In fact, starting with $a_0 = \pm 1$, we obtain two solutions $f(x) = \pm\big(2 + \frac{3}{4}x + \frac{7}{64}x^2 - \frac{21}{512}x^3 + \cdots\big)$.

3. (a) The map $f(x) \mapsto f(x^2)$ is an isomorphism from E to the subfield F.
   (b) $[E:F] = 2$; $[E:\mathbb{Q}] = [F:\mathbb{Q}] = \infty$. The infinite degree extensions have $\{1, x, x^2, x^3, \ldots\}$ as an infinite linearly independent subset. (More accurately, both of these extensions have countably infinite degree $\aleph_0$, but we accept the symbol ∞.) The extension $E \supset F$ has basis $\{1, x\}$.
   (c) As answered in class early during the semester (also on one of the handouts dealing with algebraic and transcendental extensions), the subfield $K = \mathbb{Q}(\pi) \subset \mathbb{R}$ is isomorphic to E. An explicit isomorphism $E \to K$ is given by $f(x) \mapsto f(\pi)$. The only fact that we require here about π is that it is transcendental over $\mathbb{Q}$; so in its place, we can substitute e or any other known transcendental number.

4. Let $f(x) = a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + \cdots \in \mathbb{F}_2((x))$, so that

$$1 + x^2 = (x^2 + x^4 + x^5)(a_{-2}x^{-2} + a_{-1}x^{-1} + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots).$$

   Expand the right side and solve recursively for the coefficients $a_i \in \mathbb{F}_2$ to obtain

$$\begin{aligned} f(x) &= 1x^{-2} + 0x^{-1} + 0 + 1x + 0x^2 + 1x^3 + 1x^4 + 1x^5 + \cdots \\ &= x^{-2} + x + x^3 + x^4 + x^5 + \cdots. \end{aligned}$$