Test—Monday, March 30, 2026

Instructions. The only aids allowed are a hand-held calculator and one 'cheat sheet', i.e. an 8.5″ ×11″ sheet with information written on one side in your own handwriting. No cell phones are permitted (in particular, a cell phone may not be used as a calculator). Answer as clearly and precisely as possible. *Clarity is required for full credit!* Time permitted: 50 minutes (10:00–10:50 am). Total value of questions: 100 points (plus 50 bonus points).

1. *(30 points)* Let $G = GL_2(\mathbb{R})$, the multiplicative group of all invertible 2×2 matrices with real entries.

   (a) Give an explicit example of a cyclic subgroup of order 4 in G.

   (b) Give an explicit example of a *non-cyclic* subgroup of order 4 in G.

   (c) Give an explicit example of an element of infinite order in G.

2. *(30 points)* Consider the permutations $\sigma = (12)(34)$ and $\tau = (12345)$ in S_5.

(a) What is the order of S_5?

(b) What is the order of σ?

(c) What is the order of τ?

(d) What is the (simplified) inverse of σ?

(e) What is the (simplified) inverse of τ?

(f) How many elements of S_5 commute with σ? (*Do not* list them all.)

(g) How many elements of S_5 commute with τ? (*Do not* list them all.)

(h) Simplify $\sigma^7\tau^7$.

(i) Write τ as a product of transpositions. (These transpositions will *not* be disjoint).

(j) How many elements in S_5 have the same cycle structure as σ? (*Do not* list them all.)

3. *(30 points)* Consider the multiplicative group $G = SL_2(\mathbb{Z})$ consisting of all 2×2 integer matrices $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with $a, b, c, d \in \mathbb{Z}$ having determinant $ad - bc = 1$. This group is generated by two elements $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$, a fact which *you should assume*. You are also given that there exists an isomorphism $\phi : G \to G$ satisfying

$$\phi(A) = \left[\begin{smallmatrix} 3 & 4 \\ -1 & -1 \end{smallmatrix}\right], \qquad \phi(B) = \left[\begin{smallmatrix} -5 & -9 \\ 4 & 7 \end{smallmatrix}\right].$$

Given that $C = A^2 B^{-1} = \left[\begin{smallmatrix} -1 & 2 \\ -1 & 1 \end{smallmatrix}\right]$, compute

$$\phi(C) = \quad$$

4. *(30 points)* Let n be an integer $n \geqslant 3$, and consider a multiplicative group

$$G = \{1, g, g^2, \ldots, g^{n-1},\ h, gh, g^2h, \ldots, g^{n-1}h\}$$

of order $2n$ where g has order n; h has order 2; and $hg = g^{n-1}h = g^{-1}h$.

(a) Explain how G can be viewed as the group of symmetries of a certain geometric object.

(b) Explain how G can be viewed as a permutation group of degree n. In other words, find permutations $g', h' \in S_n$ generating a subgroup of S_n isomorphic to G (and the isomorphism $G \to S_n$ maps $g \mapsto g'$, $h \mapsto h'$).

5. *(30 points)* Answer TRUE or FALSE to each of the following statements. In (a)–(e), assume that x, y are elements in a multiplicative group G.

(a) The elements xy and yx necessarily have the same order. ______*(True/False)*

(b) The subgroups $\langle x, y\rangle$ and $\langle xy, y^{-1}\rangle$ coincide, i.e. $\langle xy, y^{-1}\rangle = \langle x, y\rangle$. ______*(True/False)*

(c) The order of xy is necessarily the least common multiple of the orders of x and y. ______*(True/False)*

(d) The order of xy is necessarily the product of the orders of x and y. ______*(True/False)*

(e) If $xy = yx$, then the subgroup $\langle x, y\rangle$ is necessarily abelian. ______*(True/False)*

(f) The symmetry group of square contains the four vertices of the square. ______*(True/False)*

(g) The additive group of real numbers is cyclic, generated by the element 1. ______*(True/False)*

(h) If every element of a group G has finite order, then G must have finite order. ______*(True/False)*

(i) The additive group of integers mod 5 is a subgroup of the additive group of $\mathbb{Z}$. ______*(True/False)*

(j) Every finite group of order $n \geqslant 2$ has an abelian subgroup of order at least 2. ______*(True/False)*