Solutions to HW2

1. (a) (1735)(2648), (1638)(2547), (1836)(2745), (13)(24)(57)(68)

   (b) G has 1 element of order 1; 1 element of order 2; and 6 elements of order 4.

   (c) No, G is nonabelian. We denote $\mathbf{i} = (1234)(5678)$, $\mathbf{j} = (1537)(2846)$, $\mathbf{k} = \mathbf{ij} = (1638)(2547)$, so that $\mathbf{ji} = (1836)(2745) = \mathbf{k}^{-1}$. So G is a *quaternion* group of order 8.

2. The possible orders (i.e. 1,2,3,4,5,6) of elements in S_6 arise from eleven different cycle structures. We count the number of elements of each order by listing examples of each possible cycle structure, and the number of elements of each cycle structure.

   1 element of order 1: ();

   75 elements of order 2: $\binom{6}{2} = 15$ like (12),
   $\frac{1}{2}\binom{6}{2}\binom{4}{2} = 45$ like (12)(34), and
   $\frac{1}{6}\binom{6}{2}\binom{4}{2}\binom{2}{2} = 15$ like (12)(34)(56);

   80 elements of order 3: $2\binom{6}{3} = 40$ like (123), and
   $2\binom{6}{3}\binom{3}{3} = 40$ like (123)(456);

   180 elements of order 4: $3!\binom{6}{4} = 90$ like (1234), and
   $3!\binom{6}{4} = 90$ like (1234)(56);

   144 elements of order 5: $4!\binom{6}{5} = 144$ like (12345); and

   240 elements of order 6: $5! = 120$ like (123456), and
   $2\binom{6}{3}\binom{3}{2} = 120$ like (123)(45).

   Check: The total number of elements is $1+75+80+180+144+240 = 720 = 6! = |S_6|$.

3. (a) 1 element of order 1: ().

   (b) Each of the three pairs of opposite faces has an axis joining the centers of these faces; and each such axis has two 90° rotations, one in each direction. We count $2 \cdot 3 = 6$ rotations of 90°: (1278)(3654), (1872)(3456), (1458)(2367), (1854)(2763), (1234)(5876), (1432)(5678).

   (c) Each of the four pairs of opposite vertices is joined by an axis; and each such axis has two 120° rotations, one in each direction. We count $2 \cdot 4 = 8$ rotations of 120°: (248)(357), (284)(375), (137)(468), (173)(486), (157)(246), (175)(264), (135)(268), (153)(286).

(d) There are $3+6 = 9$ half-turns (180° rotations). Three of these are about the axes used in (b): (17)(28)(35)(46), (15)(26)(37)(48), (13)(24)(57)(68). The other six are about axes joining centers of the six pairs of opposite edges: (12)(38)(47)(56), (14)(25)(38)(67), (18)(25)(36)(47), (16)(23)(47)(58), (16)(25)(34)(78), (16)(27)(38)(45).

(e) There are $3 + 6 = 9$ reflections. Three of these are in planes midway between a pair of opposite faces: (12)(34)(56)(78), (14)(23)(58)(67), (18)(27)(36)(45). The other six are in planes containing one of the six pairs of opposite edges: (37)(48), (28)(35), (24)(57), (13)(68), (17)(46), (15)(26).

(f) The number of elements remaining is $48 - 1 - 6 - 8 - 9 - 9 = 15$. This includes

1 inversion of order 2: (16)(25)(38)(47);

8 elements of order 6: (16)(234587), (16)(278543), (143678)(25), (187634)(25), (127654)(38), (145672)(38), (123658)(47), (185632)(47). These are composites of the inversion with the 8 rotations in (c);

6 elements of order 4: (1573)(2486), (1375)(2684), (1357)(2468), (1753)(2864), (1537)(2846), (1735)(2648). These are composites of the inversion with the 6 rotations in (b).

4. (a) G has

1 element of order 1: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$;

3 elements of order 2: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; and

2 elements of order 3: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) There are six choices of isomorphism from S_3 to G, obtained by mapping a pair of distinct elements of order 2 in S_3 to distinct elements of order 2 in G (every such choice extends uniquely to an isomorphism $S_3 \to G$). One solution is

$$() \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad (12) \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}; \quad (13) \mapsto \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}; \quad (23) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$
$$(123) \mapsto \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}; \quad (132) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

5. (a) Every $g \in G$ has the form $g = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ with $g^2 = \begin{bmatrix} 1 & 2a & ac+2b \\ 0 & 1 & 2c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & ac \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
So G has

1 element of order 1: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$;

5 elements of order 2: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$; and

2 elements of order 4: $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

In fact, G is isomorphic to the dihedral group of order 8.

(b) Every $g \in G$ has the form $g = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$ with $g^3 = \begin{bmatrix} 1 & 3a & 3ac+3b \\ 0 & 1 & 3c \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
So G has 1 element of order 1, and 26 elements of order 3.