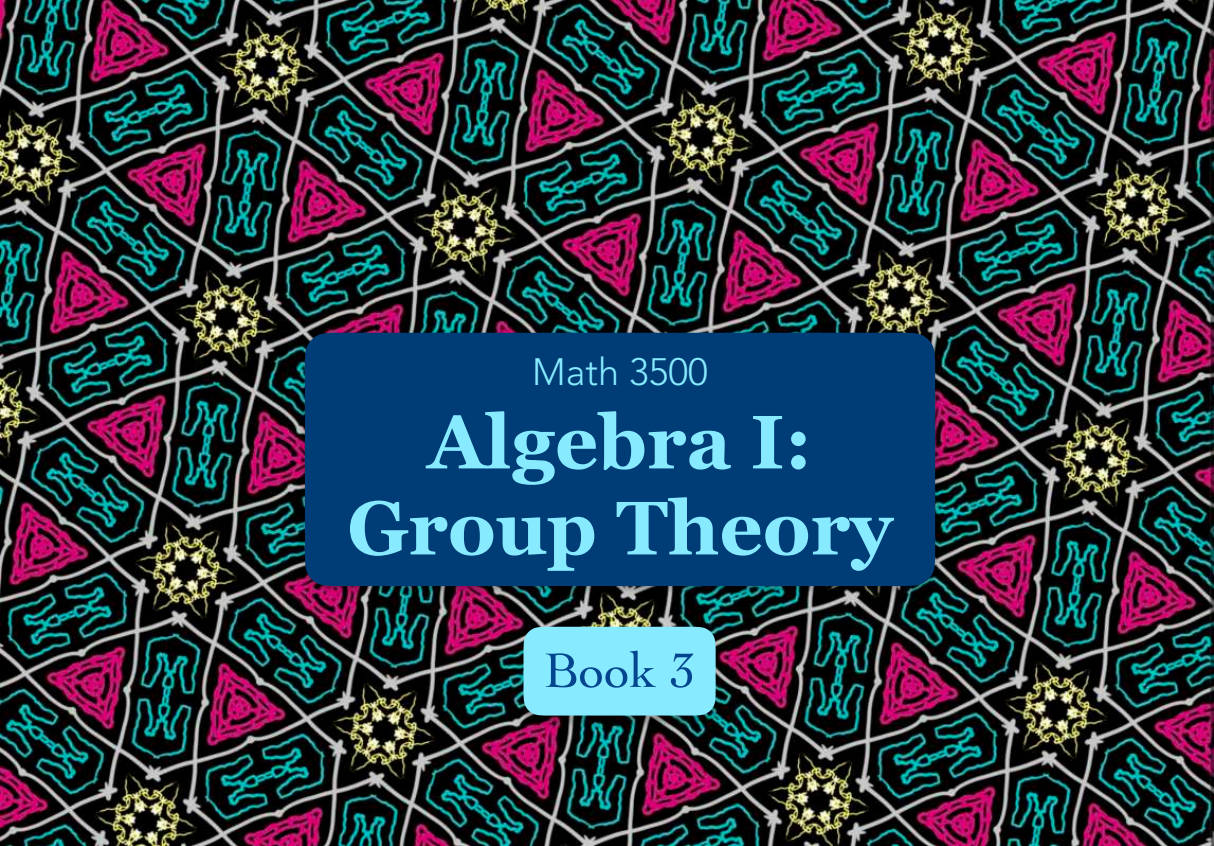
Math 3500
Algebra I:
Group Theory
Book 3

Eg. $F = \mathbb{F}_p = \{0, 1, 2, \cdots, p-1\}$ where p is a prime
(finite field of order p).

Take $n=2$ and consider the vector space $V = F^2 = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} : a, b \in F \right\}$,
an additive abelian group of order p^2.

Every homomorphism $V \to V$ is a linear transformation over the field F.

If $T: V \to V$ is a homomorphism then $T(v+w) = T(v) + T(w)$.

$T(2v) = T(v+v) = T(v) + T(v) = 2T(v)$.

$T(3v) = T(2v+v) = T(2v) + T(v) = 2T(v) + T(v) = 3T(v)$.

In fact $T(kv) = kT(v)$ for all $k \in \mathbb{F}_p$.

So $Tv = Av$ for some 2×2 matrix A over F.

There are exactly p^4 homomorphisms $V \to V$.

How many of these p^4 homomorphisms are automorphisms of V?

$(p^2-1)(p^2-p) = |GL_2(F)|$.

The Klein four-group (any group of order 4 which is not cyclic)

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

eg. $G \cong \{1, 3, 5, 7\}$ under multiplication mod 8

or $\langle (12)(34), (13)(24) \rangle < S_4$

$= \{ (), (12)(34), (13)(24), (14)(23) \}$

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$F = \mathbb{F}_2 = \{0, 1\}$ (integers mod 2)

$G \cong F^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in F \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is an additive abelian group

This is another way to look at the Klein four-group.

It has 6 automorphisms i.e. isomorphisms from the group to itself.

The group G (Klein four-group) has 16 endomorphisms (homomorphisms $G \to G$)

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

$ab = c$

$|a| = |b| = |c| = 2$

Why? To define an endomorphism T of $G = \{1, a, b, c\} = \langle a, b \rangle$

think of T as a linear transformation $T: G \to G$

there are four choices of $T(a) \in G$ i.e. $T(a) \in \{1, a, b, c\}$

" " " " $T(b) \in G$

Then $T(c) = T(ab) = T(a)T(b)$

Only 6 of these 16 homomorphisms are invertible.

How many endomorphisms does a finite cyclic group have?

Take $G = C_n = \{1, g, g^2, \dots, g^{n-1}\}$, $|g| = n$.

How many homomorphisms are there from this group to itself? Exactly n.

They are the maps $\phi_0, \phi_1, \dots, \phi_{n-1}$ where $\phi_j : G \to G$, $\phi_j(g^i) = g^{ij}$.

Note that $\phi_j(xy) = (xy)^j = x^j y^j = \phi_j(x)\phi_j(y)$ so ϕ_j is a homomorphism.

Note that $\phi_j \neq \phi_k$ for $j \neq k$ in $\{0, 1, \dots, n-1\}$ —

since $\phi_j(g) = g^j \neq \phi_k(g) = g^k$ so we have at least n different homomorphisms $C_n \to C_n$.

Conversely, suppose $\phi: C_n \to C_n$ is any homomorphism.
Then $\phi(g) = g^i \in G$, $0 \le i \le n-1$. In this case we claim $\phi = \phi_i$.

$$\phi(g^2) = \phi(gg) = \phi(g)\phi(g) = g^i g^i = g^{2i} = (g^2)^i = \phi_i(g^2)$$
$$\phi(g^3) = \phi(g^2 g) = \phi(g^2)\phi(g) = g^{2i} g^i = g^{3i} = (g^3)^i = \phi_i(g^3)$$

Inductively we get $\phi(x) = \phi_i(x)$ for all $x \in G$ i.e. $\phi = \phi_i$. □

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eg. $G = C_4 = \{1, g, g^2, g^3\}$ has four endomorphisms $\phi_0, \phi_1, \phi_2, \phi_3$ defined by

$$\phi_j(g^i) = g^{ij}$$

x	$\phi_0(x)$	$\phi_1(x)$	$\phi_2(x)$	$\phi_3(x)$
1	1	1	1	1
g	1	g	g^2	g^3
g^2	1	g^2	1	g^2
g^3	1	g^3	g^2	g

$\phi_0(g^i) = g^{0i} = g^0 = 1$
trivial homomorphism
$\phi_0(ab) = \phi_0(a)\phi_0(b)$
$1 \quad 1 \quad 1$

$\phi_1(g^i) = g^{1i} = g^i$ is the identity

$\phi_2(g^i) = g^{2i}$, $\phi_2(x) = x^2$

$\phi_3(x) = x^3$

$\phi_j(xy) = (xy)^j = x^j y^j = \phi_j(x)\phi_j(y)$

If $\psi(g^i) = g$ then $g = \psi(g^2) = \psi(gg) \neq \underbrace{\psi(g)}_{g}\underbrace{\psi(g)}_{g} = g^2$

$G = \{1, g, g^2, \dots, g^{n-1}\}$ has n homomorphisms $G \to G$, namely $\phi_k(x) = x^k$, $0 \le k \le n-1$ or $1 \le k \le n$.

How many of these are isomorphisms? (bijective)

$\phi_k : G \to G$, $x \mapsto x^k$ is one-to-one iff it's onto iff it's bijective iff $\gcd(k,n) = 1$ (k is relatively prime to n).

For $n = 12$, $\phi_k : C_{12} \to C_{12}$ is bijective iff $k \in \{1, 5, 7, 11\}$.

$\phi_3 : C_{12} \to C_{12}$ has image $\phi_3(C_{12}) = \{1, g^3, g^6, g^9\}$ ϕ_3 is neither one-to-one nor onto.

$\phi_3(1) = \phi_3(g^4) = 1$

$g \notin \phi_3(C_{12})$ $\quad g^2 = g^7$

The image of $f : G \to H$ is $f(G) = \{f(g) : g \in G\}$.

$\phi_5 : C_{12} \to C_{12}$ is onto; its image is $\{1, g^5, g^{10}, g^{15}, g^{20}, g^{25}, g^{30}, g^{35}, g^{40}, g^{45}, g^{50}, g^{55}\}$

$\qquad\qquad = g^3 \qquad = g^8 \qquad = g \qquad = g^6 \qquad = g \qquad = g^4 \qquad = g^9$

$\phi_9 : C_{12} \to C_{12}$ is not onto; $\phi_9(C_{12}) = \{1, g^9, g^6, g^3\}$ $\quad g^{60} = (g^{12})^5 = 1^5 = 1$

Euclid's Algorithm (extended form):

Let $a, b \in \mathbb{Z}$, not both zero, and let $d = \gcd(a,b)$. Then there exist integers $r, s \in \mathbb{Z}$ such that $d = ra + sb$. (That is, d is an integer linear combination of a, b).

Eg. $a = 369$, $b = 126$. We will compute $d = \gcd(a,b)$ and write d as an integer linear combination of a, b.

$369 = 2 \times 126 + 117$

$126 = 1 \times 117 + \boxed{9}$ ← $d = 9 = \gcd(a,b)$

$117 = 13 \times 9 + 0$

$369 \;\; 252 \;\; 117$

$9 = 126 - 117$

$= 126 - (369 - 2 \times 126)$

$= 3 \times 126 - 369$

$12 = 2 \times 5 + 2$

$5 = 2 \times 2 + \boxed{1} = \gcd(12,5) =$

$2 = 2 \times 1 + 0$

$1 = 5 - 2 \times 2$

$= 5 - 2(12 - 2 \times 5)$

$= 5 \times 5 - 2 \times 12$

$k = 5k \times 5 - 2k \times 12$

We want to show every element of C_{12} is the 5^{th} power of some element.

$g^k = g^{5k \times 5 - 2k \times 12} = (g^{5k})^5 (g^{12})^{-2k} = (g^{5k})^5$

$(k \in \mathbb{Z})$ ($g^{12} = 1$)

$a = 369 = 3 \times 123 = 3^2 \times 41$

$b = 126 = 3 \times 42 = 2 \times 3 \times 21 = 2 \times 3 \times 3 \times 7 = 2 \times 3^2 \times 7$.

There are n homomorphisms $\phi_k : C_n \to C_n$, $k \in \{1, 2, \cdots, n\}$ $\phi_k(x) = x^k$.

There are $\phi(n)$ isomorphisms $C_n \to C_n$, namely ϕ_k, $1 \le k \le n$, $\gcd(k,n) = 1$.

Euler's totient function $\phi(n)$ = number of integers $k \in \{1, \cdots, n\}$ such that $\gcd(k,n) = 1$.

$\phi(12) = 4$.

Sorry I'm using "ϕ" more than once.

There are exactly $\phi(n)$ elements $x \in C_n$ such that $\langle x \rangle = C_n$.

For $n = 12$, $\phi(12) = 4$ since $1, 5, 7, 11$ are the only elements $k \in \{1, 2, \cdots, 12\}$ such that $\gcd(k, 12) = 1$.

In $C_{12} = \{1, g, g^2, \cdots, g^{11}\}$, $\langle g \rangle = C_{12} = \langle g^5 \rangle = \langle g^7 \rangle = \langle g^{11} \rangle$

Suppose $f : G \to H$ is a group homomorphism.

Then $f(1_G) = 1_H$ where 1_G is the identity element of G and 1_H is the identity element of H. Eg. if $T : V \to W$ is a linear transformation then

$T(0) = 0$

(left 0: zero vector)

Proof: $f(1_G 1_G) = f(1_G) f(1_G)$. Multiply both sides on the left by $f(1_G)^{-1} \in H$ to get

$$1_H = f(1_G)^{-1} f(1_G) = \underbrace{f(1_G)^{-1} f(1_G)}_{1_H} f(1_G) = 1_H f(1_G) = f(1_G).$$ □

$$f(1_G)^{-1}(f(1_G) f(1_G)) = (f(1_G)^{-1} f(1_G)) f(1_G)$$

More generally, $|f(g)|$ divides $|g|$ for every $g \in G$ (assuming $|g| < \infty$).

If $|g| = 6$ then $|f(g)| = 1, 2, 3$ or 6.

If $|g| = 1$ then $|f(g)| = 1$ (which says $f(1_G) = 1_H$.

Proof? Note that $f(g^k) = f(\underbrace{g \cdot g \cdot g \cdots g}_{k \text{ times}}) = \underbrace{f(g) f(g) \cdots f(g)}_{k \text{ times}}$

$f(gg) = f(g)f(g)$

$f(ggg) = f(g)f(gg) = f(g)f(g)f(g)$

Now suppose $|g| = n$ and $d = |f(g)|$. We must show that $d \mid n$.

We have $n = qd + r$ for some integers q, r with $0 \le r < d$. Then

$$1 = g^n \Rightarrow \quad 1 = f(1) = f(g^n) = f(g)^n = f(g)^{qd+r} = (f(g)^d)^q f(g)^r = 1^q f(g)^r = f(g)^r.$$

By definition of the order of an element, $r = 0$, ie. $d \mid n$. □

One way in which group theory is different from linear algebra: If V, W are vector spaces and you take $v \in V$, $w \in W$ You can always find a linear transformation $T: V \to W$ such that $T(v) = w$ (unless $v = 0$ and $w \neq 0$). Recall $T(0) = 0$.

If $f: C_{12} \to C_{12}$ is a homomorphism then we cannot have $f(g^3) = g^4$ since $|g^4| = 3$ does not divide $|g^3| = 4$. Every homomorphism $f: C_{12} \to C_{12}$ must take $f(g^4) \in \{1, g^4, g^8\}$. Use ϕ_0, ϕ_1, ϕ_2 to get these.

$\{1, g, g^2, \cdots, g^{11}\}$

Careful: In S_4, there are several homomorphisms $S_4 \to S_4$.

If $f: S_5 \to S_5$ is a homomorphism, $f((12)) \in \{ (), (12), (12)(34), \dots \}$ elements of order 1 or 2

But what is an example of a homomorphism $f: S_5 \to S_5$ only.

such that $f((12)) = (12)(34)$?

You can say $f(\sigma) = \begin{cases} () & \text{if } \sigma \text{ is even} \\ (12)(34) & \text{if } \sigma \text{ is odd.} \end{cases}$ (i.e. $\sigma \in A_5$)

This is not an isomorphism.

There is an isomorphism $\phi: S_5 \to S_5$ such that $\phi((12345)) = (12453)$?

Yes.

More generally if G is a multiplicative group and $a \in G$, then we can define an isomorphism $\psi_a: G \to G$, $\psi_a(x) = axa^{-1}$ (conjugation by a).

$\psi_a(xy) = axya^{-1} = (axa^{-1})(aya^{-1}) = \psi_a(x)\psi_a(y)$ for all $x, y \in G$.

This shows that $\psi_a: G \to G$ is a homomorphism.

Why is it one-to-one? If $\psi_a(x) = \psi_a(x')$ then $axa^{-1} = ax'a^{-1}$ then $a^{-1}(axa^{-1})a = a^{-1}(ax'a^{-1})a$ so $x = x'$.

Why is it onto? For all $y \in G$, we must find $x \in G$ such that $\psi_a(x) = y$

$axa^{-1} = y$

$x = a^{-1}(axa^{-1})a = a^{-1}ya$

i.e. $\psi_a(a^{-1}ya) = y$

$\psi_a: G \to G$ is a bijection and it is a homomorphism so it is an isomorphism from G to G so it is an automorphism of G.

If G is abelian then $\psi_a(x) = axa^{-1} = aa^{-1}x = x$ i.e. ψ_a = identity.

The <u>center</u> of G is $Z(G) = \{z \in G : z \text{ commutes with every element of } G\}$

$Z(G)$ is a <u>subgroup</u> of G. If $z_1, z_2 \in Z(G)$ then $z_1 z_2 \in Z(G)$ since

$$z_1 z_2 g = z_1 g z_2 = g z_1 z_2 \quad \text{for all } g \in G.$$

Clearly $1 = 1_G$ commutes with every $g \in G$ since $1g = g = g1$, $\quad 1 \in Z(G)$.

If $z \in Z(G)$ then $z^{-1} \in Z(G)$ since for all $g \in G$,

$$zg = gz \quad \text{so} \quad z^{-1}(zg)z^{-1} = z^{-1}(gz)z^{-1} \quad \text{so} \quad gz^{-1} = z^{-1}g.$$

If $a \in Z(G)$ then $\psi_a(x) = axa^{-1} = xaa^{-1} = x$ so ψ_a = identity.

In the dihedral group of order 8, $D_8 = \{I, R, R^2, R^3, H, V, D, D'\}$ or D_4 $\quad Z(D_8) = \{I, R^2\}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $R^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$, $R^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D' = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

We have a nontrivial automorphism of D_8

$\psi_R(x) = RxR^{-1}$

$\psi_R(I) = I$

$\psi_R(-I) = -I$ $\qquad \psi_R(-I) = R(-I)R^{-1} = -IRR^{-1} = -I$

$\psi_R(R) = RRR^{-1} = R$

$\psi_R(R^3) = RR^3R^{-1} = R^{-1} = R^3$

$\psi_R(D) = D'$ $\qquad \psi_R(H) = V$

$\psi_R(D') = D$ $\qquad \psi_R(V) = H$

$\psi_D(x) = DxD^{-1} = DxD$

$\psi_D(I) = I$ $\qquad \psi_D(H) = V$

$\psi_D(-I) = -I$ $\qquad \psi_D(V) = H$

$\psi_D(R) = R^3$ $\qquad \psi_D(D) = D$

$\psi_D(R^3) = R$ $\qquad \psi_D(D') = D'$

$\psi_D(R^2) = R^2$

Four automorphisms of D_8:

$\psi_I = \psi_{R^2}$ = identity since $I, R^2 \in Z(D_8)$

$\psi_R = \psi_{R^3}$

$\psi_D = \psi_{D'}$

$\psi_H = \psi_V$

If $a, x \in G$ then we say axa^{-1} is the conjugate of x by a.
Conjugation in G is the map $x \longmapsto axa^{-1}$ for fixed $a \in G$.
We say two elements $x, y \in G$ are conjugate if $y = axa^{-1}$ for some $a \in G$.
In this case we often write $x \sim y$.

In $GL_n(\mathbb{R})$, two elements are conjugate iff they are similar.

Conjugacy (the relation $\sim$) is an equivalence relation.

D_8 has five conjugacy classes: $\{I\}, \{R^2\}, \{R, R^3\}, \{H, V\}, \{D, D'\}$.

In any group, the conjugacy classes of size 1 are $\{z\}$, $z \in Z(G)$.

$aza^{-1} = zaa^{-1} = z$

$Z(G)$ is the union of conjugacy classes of size 1 in G.

Given $a, x \in G$, $\psi_a(x) = axa^{-1}$ (the conjugate of x by a).

$\psi_a : G \to G$ which is conjugation by a.

ψ_a is an automorphism of G: ψ_a is bijective and $\psi_a(xy) = \psi_a(x)\psi_a(y)$.

Eg. Conjugation in S_n takes permutations to permutations of the same cycle structure (i.e. it preserves cycle structure).

When we count elements of S_n according to their cycle structure, we are actually counting group elements by conjugacy classes.

For $n=8$, $\sigma = (13725)(48)$, $\tau = (14)(2536)$. Conjugating σ by τ gives

$$\psi_\tau(\sigma) = \tau\sigma\tau^{-1} = \underbrace{(14)(2536)}_{\tau}\ \underbrace{(13725)(48)}_{\sigma}\ \underbrace{(14)(2635)}_{\tau^{-1}} = (18)(2)(34675) = (18)(34675)$$

Observe: σ and $\tau\sigma\tau^{-1}$ are not only the same order, they have the same cycle structure.

But conversely, if two permutations have the same cycle structure, they must be conjugate. Why?

$\sigma = (13725)(48)$ $\quad \tau = (14)(2536)$

$\tau\sigma\tau^{-1} = (46753)(18) = (18)(34675)$

Eg. Find $\tau \in S_8$ such that $\tau(135)(2746)\tau^{-1} = (1823)(457)$

$(457)(1823)$ $\quad \tau = (142)(35786)$ works.

OR $\quad \tau(135)(2746)\tau^{-1} = (1823)(457)$

$(574)(2318)$ $\quad \tau = (154)(2)(37)(68) = (154)(37)(68)$

Given $g \in G$, the centralizer of g is $C_G(g) = \{$ all elements of G that commute with $g\}$

$= \{ z \in G : zg = gz \}$

Once again, $C_G(g) \leq G$ ($C_G(g)$ is a subgroup of G).

If $z_1, z_2 \in C_G(g)$ then $z_1 z_2 \in C_G(g)$ since $(z_1 z_2)g = z_1 g z_2 = g z_1 z_2$.

$1_G \in C_G(g)$ since $1_G g = g = g 1_G$

If $z \in C_G(g)$ then $z^{-1} \in C_G(g)$ as before (back 2 pages or so).

If x and y are conjugate in G then the number of elements $a \in G$ conjugating x to y is $|C_G(x)|$.

Eg. In $G = S_8$, how many elements $\tau \in S_8$ conjugate $(1372 5)(4 8)$ to $(4675 3)(1 8)$? Same as: How many elements commute with $\sigma = (13725)(48)$. Answer: 10. in this case.

Why is $\tau\sigma\tau^{-1}$ the same as σ with all symbols in the cycle structure replaced by $i \mapsto \tau(i)$?

$\sigma = (\ldots, i, \sigma(i), \ldots)(\ldots)\cdots(\ldots)$

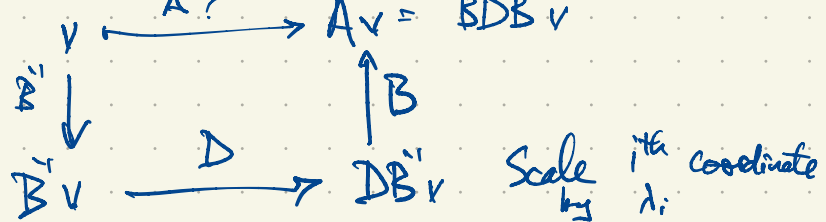
$(\tau\sigma\tau^{-1})(\tau(i)) = \tau\sigma\tau^{-1}\tau(i) = \tau(\sigma(i))$ i.e.

$\tau\sigma\tau^{-1} = (\ldots, \tau(i), \tau(\sigma(i)), \ldots)\cdots(\ldots)$

Similarly in $GL_n(F)$:

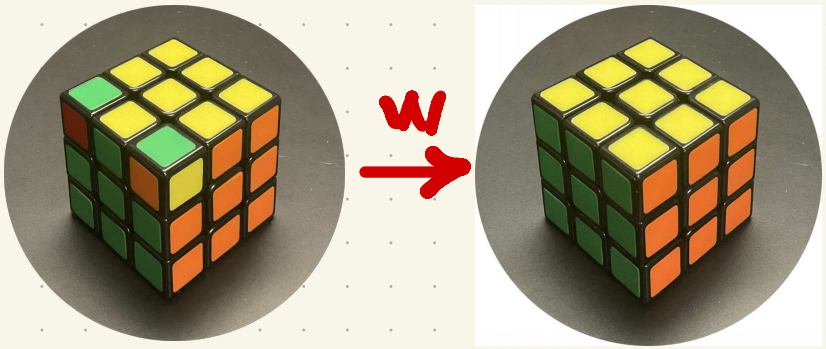
$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix}$ Scales $e_i \mapsto \lambda_i e_i$ where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ i^{th}

If $A \in GL_n(F)$ is diagonalizable (which happens over $\mathbb{C}$ "most" of the time) then A is similar (i.e. conjugate) to a diagonal matrix in $GL_n(F)$, i.e. $A = BDB^{-1}$ for some diagonal matrix D and $B \in GL_n(F)$.



Example: The group G of legal moves of Rubik's Cube has order $|G| = 43252003274489856000$ (depending a little on whether or not we count moves that move the six center pieces).

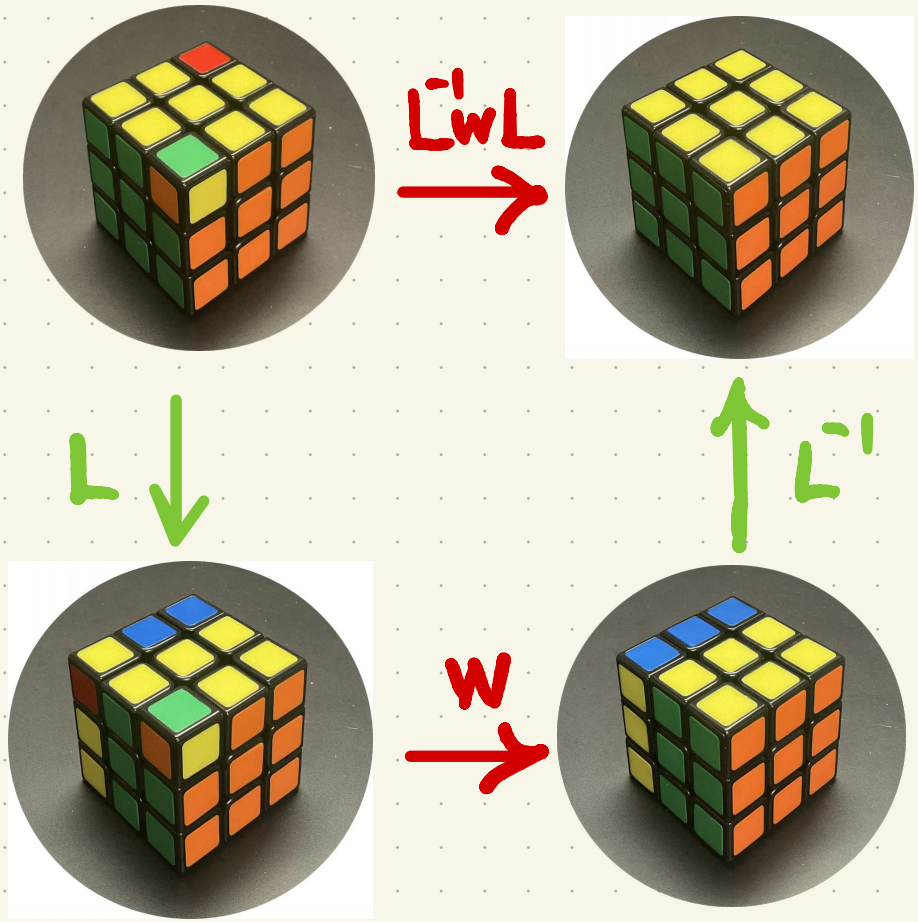
An example of conjugation in G:

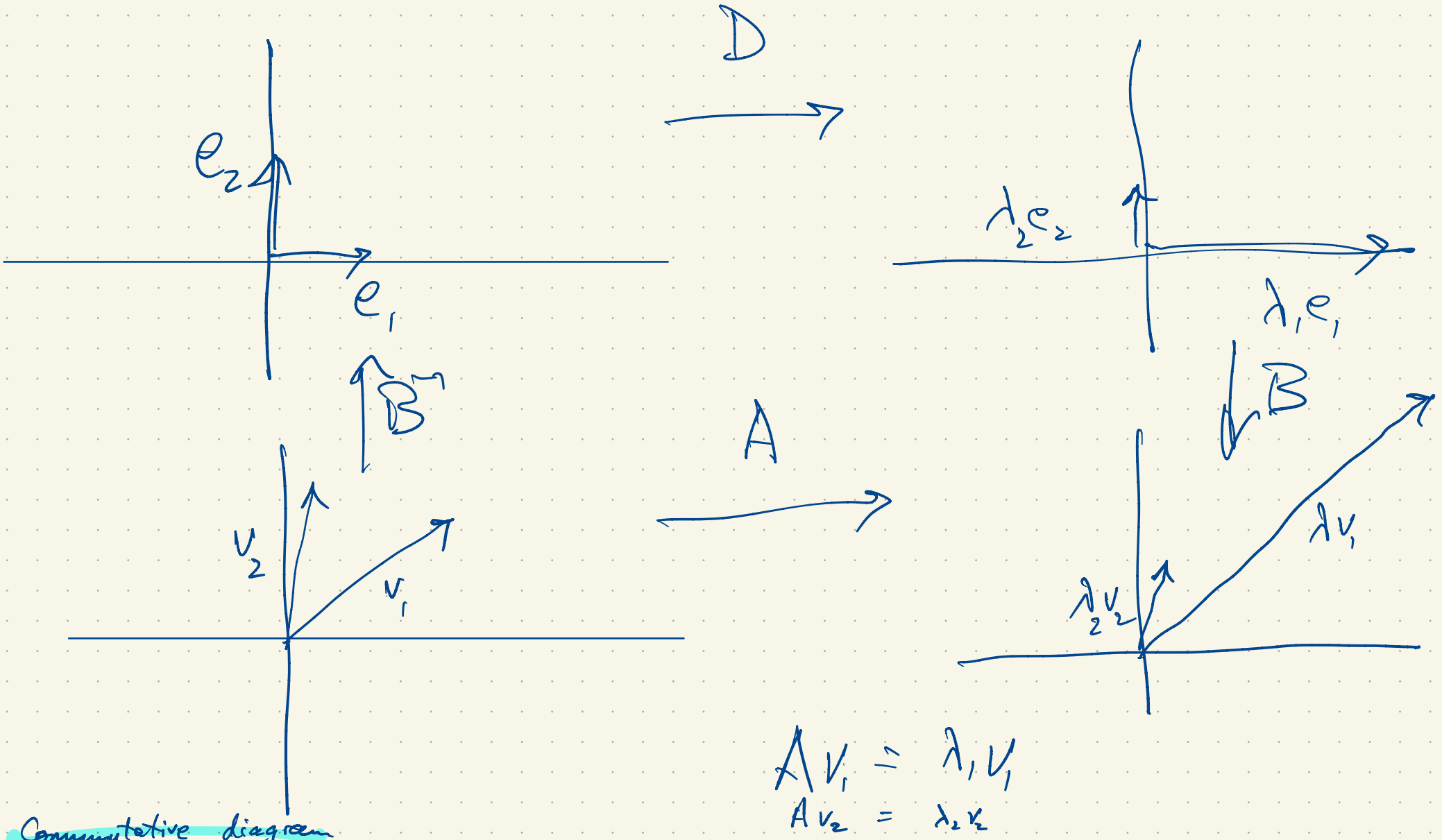


In our class, we might write

$$W = U^2 B^{-1} U^2 B U B^{-1} U B U^2 B U^2 B^{-1} U^{-1} B U^{-1} B^{-1}$$

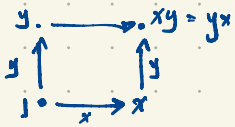
(right-to-left composition) but the standard guides for cubing would reverse this (using left-to-right composition as mathematicians prefer).





$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2$$

Commutative diagram



If G is any group then an automorphism of G is a bijection $\phi: G \to G$ such that $\phi(gh) = \phi(g)\phi(h)$ for all $g, h \in G$.

Eg. If $G = C_n = \langle g \rangle = \{1, g, g^2, \dots, g^{n-1}\}$ cyclic of order n, then there are $\phi(n)$ automorphisms of C_n where $\phi(n) = |\{k : 1 \le k \le n, \ \gcd(k, n) = 1\}|$. ϕ is Euler's totient function.

$\phi_k(x) = x^k$, $\phi_k(g^i) = g^{ki}$ is a homomorphism $C_n \to C_n$; it's an automorphism iff $\gcd(k, n) = 1$.

If $n = 6$, there are 6 homomorphisms $\phi_k: C_6 \to C_6$, $x \mapsto x^k$.

$\phi_1(x) = x$ identity

$\phi_2(x) = x^2$ is not one-to-one since $\phi_2(g^3) = (g^3)^2 = g^6 = 1 = \phi_2(1)$

$\phi_3(x) = x^3$ is not one-to-one since $\phi_3(g^2) = (g^2)^3 = g^6 = 1 = \phi_3(1) = \phi_3(g^4) = g^{12} = 1$

$\phi_4(x) = x^4$ is not one-to-one since $\phi_4(g^3) = (g^3)^4 = g^{12} = 1 = \phi_4(1)$

$\phi_5(x) = x^5 = x^{-1}$ is one-to-one and onto.

x	$\phi_5(x)$
1	1
g	g^5
g^2	g^4
g^3	g^3
g^4	g^2
g^5	g

Eg. $\mathrm{Aut}(G) = \{\text{all automorphisms of } G\}$ is a group (not to be confused with G), the automorphism group of G.

$G = C_n$ $\quad \mathrm{Aut}(C_6) = \{\phi_1, \phi_5\} \cong C_2$ (ϕ_1: identity)

$\circ$	ϕ_1	ϕ_5
ϕ_1	ϕ_1	ϕ_5
ϕ_5	ϕ_5	ϕ_1

eg. $\phi_5\phi_5(x) = (x^5)^5 = x^{25} = x^{24} \cdot x = 1 \cdot x = x = \phi_1(x)$

	1	g	g^2	g^3	g^4	g^5
1	1	g	g^2	g^3	g^4	g^5
g	g	g^2	g^3	g^4	g^5	1
g^2	g^2	g^3	g^4	g^5	1	g
g^3	g^3	g^4	g^5	1	g	g^2
g^4	g^4	g^5	1	g	g^2	g^3
g^5	g^5	1	g	g^2	g^3	g^4

this is not that! $G \ncong \mathrm{Aut}\, G$.

If $a \in G$ then $\phi_a(x) = axa^{-1}$ (conjugation by $a \in G$) is an inner automorphism of G.
This gives lots of examples of automorphisms of G if G is nonabelian.

Abelian groups tend to have lots of automorphisms but they're not inner (except for the identity)
$id: G \to G$, $id(g) = g$. $id(gh) = gh = id(g)\,id(h)$ $id \in \mathrm{Aut}\,G$ (not 1_G)
If F is any field then $V = F^n$ is usually thought of as a vector space of dimension n over F.
Ignoring scalar multiplication, this is a group under $+$ with identity element the zero vector.
If $F = \mathbb{F}_p = \{0, 1, 2, \dots, p-1\}$. In this case F is a cyclic group under addition and automorphisms of this group are scalar multiplication by nonzero elements of F.

$\mathrm{Aut}(F^n) \geq GL_n(F)$
Each $A \in GL_n(F)$ gives an automorphism of F^n, $A: v \to Av$
This is a homomorphism $F^n \to F^n$ since $A(v+w) = Av + Aw$
Since A is invertible, $A: F^n \to F^n$ is bijective. The inverse of $A: v \to Av$ is
$A^{-1}: v \to A^{-1}v$.
Sometimes F^n has more automorphisms than these but often not.
An inner automorphism of F^n is a function of the form $\phi_a: v \to a + v + (-a) = v$. $a, v \in F^n$

$\phi_a(v) = v$ i.e. $\phi_a = id$ for all $a \in V$.

Eg. $\mathrm{Aut}(S_n) \cong S_n$ if $n \neq 6$. S_6 has outer automorphisms (automorphisms that are not inner)
There is $f \in \mathrm{Aut}(S_6)$ such that $f((12)) = (12)(34)(56)$ and this cannot be an inner automorphism.

Theorem There is a homomorphism $G \to \mathrm{Aut}\,G$ defined by $a \mapsto \phi_a$. Shoe-Sock Theorem
Proof $\phi_{ab} = \phi_a \phi_b$ because $(\phi_a \circ \phi_b)(g) = \phi_a(\phi_b(g)) = a(bgb^{-1})a^{-1} = (ab)g(b^{-1}a^{-1}) = (ab)g(ab)^{-1} = \phi_{ab}(g)$
for all $a, b, g \in G$.

$GL_4(\mathbb{F}_2) \cong A_8$. of order 20160 $\mathrm{Aut}(GL_4(\mathbb{F}_2)) \cong S_8$. of order 40320.

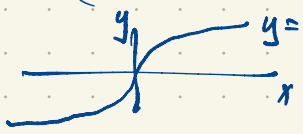
All automorphisms of $SL_2(\mathbb{R})$ are inner. (I think)
For $n>2$, there are outer automorphisms of $SL_n(\mathbb{R})$.

Let's explain starting with $GL_1(\mathbb{R})$.
$\mathrm{Aut}\, GL_n(\mathbb{R})$ includes inner automorphisms. $\phi_B(A) = BAB^{-1}$
$GL_2(\mathbb{R})$ has an automorphism $\phi(A) = (\det A)A$.

$$\phi(AB) = \det(AB)\,AB = (\det A)(\det B)\,AB = (\det A)A \cdot (\det B)B = \phi(A)\,\phi(B).$$

Why is $\phi : GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ bijective?
Suppose $\phi(A) = \phi(B)$ i.e. $(\det A)A = (\det B)B$.

$$(\det A)^3 = (\det B)^3 \Rightarrow \det A = \det B \Rightarrow A = B.$$



Why is ϕ onto?
Given $A \in GL_2(\mathbb{R})$, find $X \in GL_2(\mathbb{R})$ such that $\phi(X) = A$

$$(\det X)X = A$$
$$(\det X)^3 = \det A$$
$$\det X = (\det A)^{1/3}$$

$\det(cA) = c^2 \det A$

$$\det c\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \det\begin{bmatrix} cx & cy \\ cz & cw \end{bmatrix} = c^2(xw - yz) = c^2 \det A$$

Try $X = (\det A)^k A$

$$\phi(X) = (\det X)X = (\det A)^{2k+1}(\det A)A = \underline{(\det A)^{3k+1} A}$$